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Representations of quantum bicrossproduct algebras

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Abstract
We present a method to construct induced representations of quantum algebras which have a bicrossproduct structure. We apply this procedure to some quantum kinematical algebras in $(1+1)$ dimensions with this kind of structure: null-plane quantum Poincaré algebra, non-standard quantum Galilei algebra and quantum κ-Galilei algebra.

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1. Introduction

In a recent paper [1] we developed a method to construct induced representations of quantum algebras mainly based on the concepts of module and duality. Since by dualization objects such as modules and comodules can be seen as equivalent, then we have not only regular and induced representations but also coregular and coinduced representations. The main result was the possibility of constructing coregular and coinduced representations of a Hopf algebra $U_q(\mathfrak{g})$ when dual bases of this and its dual $Fun_q(G)$ (or $F_q(G)$) are known, with $\mathfrak{g}$ being the Lie algebra of a Lie group G.

Now we present a particular study of the induction method for those quantum algebras having a bicrossproduct structure, which is a generalization of the idea of a semi-direct product of Lie groups to the quantum case. This kind of structure of a semi-direct product is well known in physics where many interesting groups have it, such as Euclidean, Galilei and Poincaré. The corresponding quantum Lie algebras inherit the 'semi-direct' structure in the algebra sector and the algebra of functions also has a semi-direct product structure in the coalgebra sector. These ideas were generalized by Molnar [2] with the notion of smash product or, more recently, by Majid with that of bicrossproduct [3–6].

The quantum counterparts of the above-mentioned groups and algebras are related to the symmetries of the physical space–time in a noncommutative framework. The study of these quantum symmetries and their representations generalizes the well-known and fruitful program started by Wigner in 1939 [7] inside the perspective of noncommutative geometry [8], which in the last few years has increased its applications in physics (see, for instance, [9] and references therein).

In this paper, we continue the analysis of the theory of induced representations but now refer to Hopf algebras with a bicrossproduct structure, whose first factor is cocommutative and the second factor commutative.

Induced representations (or induced modules in the algebraic jargon) is a well-known technique [10] that permits us to construct representations of an algebra using, as starting point, representations of a subalgebra. The successful implementation of these techniques in the Lie group context [11–13] requires the introduction of some appropriate algebra (e.g. the algebra of the group functions, the enveloping algebra of the Lie algebra, etc) associated to the group. In our case the algebra structure is provided *ab initio* by the Hopf algebra framework and, hence, the induction method can start in a natural way.

Our main objective is the description of the representations induced by characters of the commutative sector. We want to avoid the problems derived by the use of pairs of dual bases and to open new ways which allow us to perform some computations, whose difficulty increases with the number of generators. Although the first results are also obtained using dual bases, they show the existence of an underlying structure connected with some one-parameter flows defined by the cocommutative sector over the manifold related to the commutative factor. The nature of this structure will be clear after an adequate re-interpretation of the factors of the bicrossproduct. More explicitly, the cocommutative factor will be seen as the enveloping algebra of a certain Lie algebra but the commutative factor will be identified with the algebra of functions of another Lie group. In this way, the action, defining a part of the bicrossproduct structure in the original Hopf algebra, is the result of translating to the algebra of functions the action of a Lie group over another Lie group.

A crucial point in our approach is to describe the Hopf algebra substituting the monomial bases by elements which are product of an element of the group associated to the first factor times a function belonging to the second one. This factorization allows us to prove the theorems 4.1 and 4.3 that are the cornerstone of this paper. The first theorem characterizes four regular modules associated to a bicrossproduct Hopf algebra in terms of the regular actions of its components and the action, mentioned above, associated with the bicrossproduct structure. The second theorem describes the construction of representations induced by characters of the Abelian sector and classifies the equivalence classes of the induced representations in terms of the orbits associated to the action of a certain group. Moreover, a $*$-structure is introduced in such a way that the induced representations are unitary.

The induction procedure, such as it has been formulated by us, has an algebraic characteristic since we use objects such as modules, comodules, etc, which are the appropriate tools to work with the algebraic structures exhibited by the quantum groups and algebras [1, 14–16]. In [17–19] we have presented some results based on this work.

In the literature we can find some works about induced representations of quantum groups. Dobrev [20, 21] has developed a method to construct representations of quantum groups similar, in some sense, to ours, i.e. both methods emphasize the dual case, closer to the classical one, and the representations are constructed in the algebra sector. In [22–26] Mackey's theory [11, 12] for Lie groups has been extended to quantum groups but constructing corepresentations, i.e. representations of the coalgebra sector.

The paper is organized as follows. Section 2 is devoted to a review of the main ideas and concepts that we use throughout the paper, such as module, comodule, module algebra, bicrossproduct, etc. The final part of this section presents original results showing how pairs of dual bases and $*$-structures over bicrossproduct Hopf algebras can be obtained starting from those of their factors. The first results about induced representations of quantum algebras with a bicrossproduct structure are presented in section 3. We obtain the representations making use of pairs of dual bases. In section 4 we study the induction problem taking into account the relationship between modules and representations. The concept of regular co-space allows us to obtain deeper results from a geometric point of view. In section 5 we obtain the induced representations of some kinematical quantum algebras making use of the method developed in the previous sections. We end with some comments and conclusions.

2. Mathematical preliminaries

Let $H = (V; m\eta; \Delta\epsilon; S)$ be a Hopf algebra with underlying vector space V over the field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$), multiplication $m : H \otimes H \to H$, coproduct $\Delta : H \to H \otimes H$, unit $\eta : \mathbb{K} \to H$, counit $\epsilon : H \to \mathbb{K}$ and antipode $S : H \to H$.

A Hopf algebra can be considered as a bialgebra with an antilinear map S, and a bialgebra can be seen as composed by two 'substructures' or 'sectors' (the algebra sector (V, m, η) and the coalgebra sector (V, Δ, ϵ)) with some compatibility conditions [27].

On the other hand, the algebras considered in this paper are finitely generated although they are infinite dimensional. For this reason the following multi-index notation is very useful [1]. Let us suppose that A is an algebra generated by the elements $(a_1, a_2, \dots, a_r)$ and the ordered monomials,

$$a^n := a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r} \in A \qquad n = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r \tag{2.1}$$

form a basis of the underlying linear space to A. An arbitrary product of generators of A is written in a normal ordering if it is expressed in terms of the basis $(a^n)_{n\in\mathbb{N}^r}$. In some cases, we use the notation $\left(a_n := a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}\right)_{n\in\mathbb{N}^r}$. For $0 = (0, \dots, 0) \in \mathbb{N}^n$ we have $a^0 \equiv 1_A$. Multi-factorials and multi-deltas are defined by

$$l! = \prod_{i=1}^{n} l_i! \qquad \delta_l^m = \prod_{i=1}^{n} \delta_{l_i}^{m_i}. \tag{2.2}$$

2.1. Duality

It is well known that the dual object of V is defined as the vector space of its linear forms, i.e., $V^* = \mathcal{L}(V, \mathbb{K})$. Hence, if (V, m, η) is a finite algebra it is natural to define the dual object as (V^*, m^*, η^*) obtaining a coalgebra, and vice versa. However, in the infinite-dimensional case the spaces $(V \otimes V)^*$ and $V^* \otimes V^*$ are not isomorphic and some problems appear with the coproduct as dual of the multiplication map. The concept of pairing solves these difficulties.

A pairing between two Hopf algebras [27], H and H', is a bilinear mapping $\langle\cdot,\cdot\rangle$: $H \times H' \to \mathbb{K}$ that verifies the following properties:

$$\begin{aligned} &\langle h, m'(h' \otimes k')\rangle = \langle \Delta(h), h' \otimes k'\rangle \qquad &&\langle h, 1_{H'}\rangle = \epsilon(h) \\ &\langle h \otimes k, \Delta'(h')\rangle = \langle m(h \otimes k), h'\rangle \qquad &&\epsilon'(h') = \langle 1_H, h'\rangle \\ &\langle h, S'(h')\rangle = \langle S(h), h'\rangle. \end{aligned} \tag{2.3}$$

We remark that $\langle h \otimes k, h' \otimes k'\rangle = \langle h, h'\rangle\langle k, k'\rangle$.

The pairing is said to be left (right) non-degenerate if $[\langle h, h'\rangle = 0, \forall h' \in H'] \Rightarrow h = 0$ ($[\langle h, h'\rangle = 0, \forall h \in H] \Rightarrow h' = 0$). If the pairing is simultaneously left and right non-degenerate we simply say that it is non-degenerate.

The triplet $(H, H', \langle\cdot,\cdot\rangle)$ composed by two Hopf algebras and a non-degenerate pairing will be called a 'non-degenerate triplet'.

The bases (h^m) of H and (h'_n) of H' are said to be dual with respect to the non-degenerate pairing if

$$\langle h^m, h'_n\rangle = c_n\delta_n^m \qquad c_n \in \mathbb{K} - \{0\}. \tag{2.4}$$

The map $f^\dagger : H' \to H'$ implicitly defined in terms of the map $f : H \to H$ by

$$\langle h, f^\dagger(h')\rangle = \langle f(h), h'\rangle \tag{2.5}$$

is called the adjoint map of f with respect to the non-degenerate pairing.

2.2. Modules and comodules

Let us consider the triad (V, α, A), where A is an associative $\mathbb{K}$-algebra with unit, V is a $\mathbb{K}$-vector space and α a linear map, $\alpha : A \otimes_{\mathbb{K}} V \to V$, called action and denoted by $a \triangleright v = \alpha(a \otimes v)$. We say that (V, α, A) (or $(V, \triangleright, A)$) is a left A-module if the following two conditions are verified:

$$a \triangleright (b \triangleright v) = (ab) \triangleright v \qquad 1 \triangleright v = v \qquad \forall a, b \in A \qquad \forall v \in V. \tag{2.6}$$

A morphism of left A-modules, $(V, \triangleright, A)$ and $(V', \triangleright', A)$, is a linear map, $f : V \to V'$, equivariant in respect of the action, i.e.,

$$f(a \triangleright v) = a \triangleright' f(v) \qquad \forall a \in A \quad \forall v \in V. \tag{2.7}$$

Dualizing the concept of the A-module, we obtain the concept of a comodule. Thus, if C is an associative $\mathbb{K}$-coalgebra with counit, V a $\mathbb{K}$-vector space and $\beta : V \to C \otimes_{\mathbb{K}} V$ a linear map that will be called coaction and denoted by $v \blacktriangleleft = \beta(v) = v^{(1)} \otimes v^{(2)}$, the triad (V, β, C) (or $(V, \blacktriangleleft, C)$) is said to be a left C-comodule if the following axioms are verified

$$v^{(1)}{}_{(1)} \otimes v^{(1)}{}_{(2)} \otimes v^{(2)} = v^{(1)} \otimes v^{(2)}{}_{(1)} \otimes v^{(2)}{}_{(2)} \qquad \epsilon(v^{(1)})v^{(2)} = v \qquad \forall v \in V \tag{2.8}$$

where the coproduct of the elements of C is symbolically written as $\Delta(c) = c_{(1)} \otimes c_{(2)}$.

A linear map $f : V \to V'$ between two C-comodules, $(V, \blacktriangleleft, C)$ and $(V', \blacktriangleleft', C)$ is a morphism if

$$v^{(1)} \otimes f(v^{(2)}) = f(v)^{(1)'} \otimes f(v)^{(2)'} \qquad \forall v \in V. \tag{2.9}$$

Similarly right A-modules and right C-comodules are defined.

2.3. Module algebras

When a bialgebra acts or coacts on a vector space equipped with an additional algebra, coalgebra or bialgebra structure [2, 28] it is natural to demand some compatibility relations for the action. In the following, B and B' denote bialgebras, A an algebra and C a coalgebra.

The left module $(A, \triangleright, B)$ is said to be a B-module algebra if m_A and η_A are morphisms of B-modules, i.e. if

$$b \triangleright (aa') = (b_{(1)} \triangleright a)(b_{(2)} \triangleright a') \qquad b \triangleright 1 = \epsilon(b)1 \qquad \forall b \in B, \forall a, a' \in A. \tag{2.10}$$

Substituting algebra by coalgebra, we obtain the structure of module coalgebra. In this case, the left B-module $(C, \triangleright, B)$ is a B-module coalgebra if Δ_C and ϵ_C are morphisms of B-modules, i.e. if

$$(b \triangleright c)_{(1)} \otimes (b \triangleright c)_{(2)} = (b_{(1)} \triangleright c_{(1)}) \otimes (b_{(2)} \triangleright c_{(2)}) \qquad \epsilon_C(b \triangleright c) = \epsilon_B(b)\epsilon_C(c) \quad \forall b, c \in B.$$

Dualizing these structures two new structures are obtained. The left B-comodule $(C, \blacktriangleleft, B)$ is said to be a B-comodule coalgebra if Δ_C and ϵ_C are morphisms of B-comodules, i.e.,

$$c^{(1)} \otimes c^{(2)}{}_{(1)} \otimes c^{(2)}{}_{(2)} = c_{(1)}{}^{(1)}c_{(2)}{}^{(1)} \otimes c_{(1)}{}^{(2)} \otimes c_{(2)}{}^{(2)} \qquad c^{(1)}\epsilon_C(c^{(2)}) = (\eta_B \circ \epsilon_C)(c).$$

The left B-comodule $(A, \blacktriangleleft, B)$ is a B-comodule algebra if m_A and η_A are morphisms of B-comodules. Explicitly

$$(aa')_{(1)} \otimes (aa')_{(2)} = a_{(1)}a'_{(1)} \otimes a_{(2)}a'_{(2)} \qquad 1_A \blacktriangleleft = 1_B \otimes 1_A. \tag{2.11}$$

The triad $(B', \triangleright, B)$ is a left B-module bialgebra if simultaneously it is a B-module algebra and a B-module coalgebra; $(B', \blacktriangleleft, B)$ is a left B-comodule bialgebra if simultaneously it is a B-comodule algebra and a B-comodule coalgebra.

The corresponding versions for the right are defined in an analogous manner.

By regular module (comodule) we understand an A-module (C-comodule) whose vector space is the underlying vector space of the algebra A (coalgebra C). The action (coaction) is defined by means of the algebra product (coalgebra coproduct).

For instance, on the regular A-modules $(A, \triangleright, A)$ and $(A, \triangleleft, A)$ the actions are, respectively,

$$a \triangleright a' = aa' \qquad a' \triangleleft a = a'a. \tag{2.12}$$

If B is a bialgebra, the regular B-module $(B, \triangleright, B)$ whose 'regular' action is defined by

$$b \triangleright b' = bb', \tag{2.13}$$

is a module coalgebra. The module $(B^*, \triangleleft, B)$, obtained by dualization, is a module algebra with the 'regular' action

$$\varphi \triangleleft b = \langle \varphi_{(1)}, b \rangle \varphi_{(2)} \qquad b \in B \quad \varphi \in B^* \tag{2.14}$$

which is also called regular module. The comodule versions can be easily obtained by the reader.

2.4. Bicrossproduct structure

The concepts of module algebra and comodule coalgebra allow us to describe in a suitable way 'semi-direct' structures [2, 28] as we shall see later.

Let H be a bialgebra and $(A, \triangleleft, H)$ a right A-module algebra. The expression

$$(h \otimes a)(h' \otimes a') = hh'_{(1)} \otimes (a \triangleleft h'_{(2)})a' \tag{2.15}$$

defines an algebra structure over $H \otimes A$, denoted by $H \ltimes A$ and called a semi-direct product at the right (or simply a right semi-direct product) of A and H.

The 'left' version is as follows. Let $(A, \triangleright, H)$ be a left A-module algebra. A structure of algebra over $H \otimes A$, denoted by $A \rtimes H$ and called the left semi-direct product of A and H, and is defined by means of

$$(a \otimes h)(a' \otimes h') = a(h_{(1)} \triangleright a') \otimes h_{(2)}h'. \tag{2.16}$$

The corresponding dual structures are constructed in the following way. Let $(C, \blacktriangleleft, H)$ be a left C-comodule coalgebra. A coalgebra structure over $C \otimes H$, denoted by $C \mathbin{>\!\!\blacktriangleleft} H$ and called the left semi-direct product, is obtained if

$$\begin{aligned} \Delta(c \otimes h) &= c_{(1)} \otimes c_{(2)}{}^{(1)} h_{(1)} \otimes c_{(2)}{}^{(2)} \otimes h_{(2)} \\ \epsilon(c \otimes h) &= \epsilon_C(c)\epsilon_H(h). \end{aligned} \tag{2.17}$$

When $(C, \blacktriangleright, H)$ is a right C-comodule coalgebra, the expressions

$$\begin{aligned} \Delta(h \otimes c) &= h_{(1)} \otimes c_{(1)}{}^{(1)} \otimes h_{(2)} c_{(1)}{}^{(2)} \otimes c_{(2)} \\ \epsilon(h \otimes c) &= \epsilon_C(h)\epsilon_H(c) \end{aligned} \tag{2.18}$$

characterize a coalgebra structure over $C \otimes H$ denoted by $C \blacktriangleright\!\!< H$ and called the right semi-direct product of C and H.

Let K and L be two bialgebras, such that $(L, \triangleleft, K)$ is a right K-module algebra and $(K, \blacktriangleleft, L)$ a left L-comodule coalgebra. The tensor product $K \otimes L$ is equipped simultaneously with the semi-direct structures of algebra $K \triangleright\!\!< L$ and coalgebra $K >\!\!\blacktriangleleft L$. If the following compatible conditions are verified

$$\begin{array}{ll} \epsilon(l \triangleleft k) = \epsilon(l)\epsilon(k) & \Delta(l \triangleleft k) = (l_{(1)} \triangleleft k_{(1)}){k_{(2)}}^{(1)} \otimes l_{(2)} \triangleleft {k_{(2)}}^{(2)} \\ 1 \blacktriangleleft = 1 \otimes 1 & (kk') \blacktriangleleft = (k^{(1)} \triangleleft k'_{(1)}){k'_{(2)}}^{(1)} \otimes k^{(2)}{k'_{(2)}}^{(2)} \\ {k_{(1)}}^{(1)}(l \triangleleft k_{(2)}) \otimes {k_{(1)}}^{(2)} = (l \triangleleft k_{(1)}){k_{(2)}}^{(1)} \otimes {k_{(2)}}^{(2)} & \end{array} \tag{2.19}$$

then $K \triangleright\!\!< L$ and $K >\!\!\blacktriangleleft L$ determine a bialgebra called (right–left) bicrossproduct and denoted by $K \triangleright\!\!\blacktriangleleft L$.

If K and L are two Hopf algebras, then $K \triangleright\!\!\blacktriangleleft L$ also has an antipode given by

$$S(k \otimes l) = (1 \otimes S(k^{(1)}l))(S(k^{(2)}) \otimes 1). \tag{2.20}$$

On the other hand, let $\mathcal{K}$ and $\mathcal{L}$ be two bialgebras and $(\mathcal{L}, \triangleright, \mathcal{K})$ and $(\mathcal{K}, \blacktriangleright, \mathcal{L})$ a left K-module algebra and a right L-comodule coalgebra, respectively, verifying the compatibility conditions

$$\begin{array}{l} \varepsilon(\lambda \triangleright \kappa) = \varepsilon(\lambda)\varepsilon(\kappa) \\ \Delta(\lambda \triangleright \kappa) \equiv (\lambda \triangleright \kappa)_{(1)} \otimes (\lambda \triangleright \kappa)_{(2)} = \left({\lambda_{(1)}}^{(1)} \triangleright \kappa_{(1)}\right) \otimes {\lambda_{(1)}}^{(2)}(\lambda_{(2)} \triangleright \kappa_{(2)}) \\ \blacktriangleright (1) = 1 \otimes 1 \\ \blacktriangleright (\kappa\kappa') = {\kappa_{(1)}}^{(1)}\kappa'^{(1)} \otimes {k_{(1)}}^{(2)}(\kappa_{(2)} \triangleright \kappa'^{(2)}) \\ {\lambda_{(2)}}^{(1)} \otimes (\lambda_{(1)} \triangleright \kappa){\lambda_{(2)}}^{(2)} = {\lambda_{(1)}}^{(1)} \otimes {\lambda_{(1)}}^{(2)}(\lambda_{(2)} \triangleright \kappa). \end{array} \tag{2.21}$$

Then $\mathcal{L} >\!\!\triangleleft \mathcal{K}$ and $\mathcal{L} \blacktriangleright\!\!< \mathcal{K}$ determine a bialgebra called (left–right) bicrossproduct denoted by $\mathcal{L} \blacktriangleright\!\!\triangleleft \mathcal{K}$.

If $\mathcal{K}$ and $\mathcal{L}$ are two Hopf algebras then $\mathcal{L} \blacktriangleright\!\!\triangleleft \mathcal{K}$ has an antipode defined by

$$S(\lambda \otimes \kappa) = (1 \otimes S\kappa^{(1)})(S(\lambda\kappa^{(2)}) \otimes 1). \tag{2.22}$$

Note that both bicrossproduct structures are related by duality. Effectively, it can be proven that, if K and L are two finite-dimensional bialgebras, and the right K-module algebra $(L, \triangleleft, K)$ and the left L-comodule coalgebra $(K, \blacktriangleleft, L)$ verify the conditions (2.19), then $(K \triangleright\!\!\blacktriangleleft L)^* = K^* \blacktriangleright\!\!\triangleleft L^*$.

2.5. Star structures over bicrossproduct Hopf algebras

The following results show how to construct dual bases and $*$-structures over Hopf algebras with the structure of bicrossproduct when the corresponding objects of the factors of the bicrossproduct are known [15].

Theorem 2.1. *Let $H = K \triangleright\!\!\blacktriangleleft L$ be a Hopf algebra with a bicrossproduct structure, and $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ non-degenerate pairings for the pairs (K, K^*) and (L, L^*), respectively. Then the expression*

$$\langle kl, \kappa\lambda\rangle = \langle k, \kappa\rangle_1 \langle l, \lambda\rangle_2 \tag{2.23}$$

defines a non-degenerate pairing between H and H^*.

Proof. Firstly note that

$$\langle 1, \kappa\lambda\rangle = \langle 1_K \otimes 1_L, \kappa\lambda\rangle = \langle 1_K, \kappa\rangle\langle 1_L, \lambda\rangle = \epsilon(\kappa)\epsilon(\lambda) = (\epsilon \otimes \epsilon)(\kappa \otimes \lambda) = \epsilon(\kappa\lambda). \tag{2.24}$$

On the other hand

$$\begin{aligned}\langle kl, (\kappa\lambda)(\kappa'\lambda')\rangle &= \langle kl, \kappa(\lambda_{(1)} \triangleright \kappa')\lambda_{(2)}\lambda'\rangle = \langle k, \kappa(\lambda_{(1)} \triangleright \kappa')\rangle_1 \langle l, \lambda_{(2)}\lambda'\rangle_2\\ &= \langle k_{(1)}, \kappa\rangle_1 \langle k_{(2)}, \lambda_{(1)} \triangleright \kappa'\rangle_1 \langle l_{(1)}, \lambda_{(2)}\rangle_2 \langle l_{(2)}, \lambda'\rangle_2\\ &= \langle k_{(1)}, \kappa\rangle_1 \langle \tau(k_{(2)} \blacktriangleleft), \kappa' \otimes \lambda_{(1)}\rangle \langle l_{(1)}, \lambda_{(2)}\rangle_2 \langle l_{(2)}, \lambda'\rangle_2\\ &= \langle k_{(1)}, \kappa\rangle_1 \left\langle {k_{(2)}}^{(2)}, \kappa'\right\rangle_1 \left\langle {k_{(2)}}^{(1)}, \lambda_{(1)}\right\rangle_2 \langle l_{(1)}, \lambda_{(2)}\rangle_2 \langle l_{(2)}, \lambda'\rangle_2\\ &= \langle k_{(1)}, \kappa\rangle_1 \left\langle {k_{(2)}}^{(2)}, \kappa'\right\rangle_1 \left\langle {k_{(2)}}^{(1)} l_{(1)}, \lambda\right\rangle_2 \langle l_{(2)}, \lambda'\rangle_2\\ &= \left\langle k_{(1)} {k_{(2)}}^{(1)} l_{(1)}, \kappa\lambda\right\rangle \left\langle {k_{(2)}}^{(2)} l_{(2)}, \kappa'\lambda'\right\rangle\\ &= \langle \Delta(kl), (\kappa\lambda) \otimes (\kappa'\lambda')\rangle. \end{aligned} \tag{2.25}$$

Similarly for the identities

$$\langle kl, 1\rangle = \epsilon(kl) \qquad \langle (kl)(k'l'), \kappa\lambda\rangle = \langle (kl) \otimes (k'l'), \Delta(\kappa\lambda)\rangle. \tag{2.26}$$

Hence, it is proven that $\langle\cdot,\cdot\rangle$ is a bialgebra pairing. The pairing is non-degenerate. Effectively, fixing a basis $(l_i)_{i\in I}$ of L, the coproduct can be written as

$$\Delta(h) = \sum_{i\in I} a_i(h) \otimes l_i \tag{2.27}$$

with $a_i : H \to K$. Let us suppose that

$$\langle h, \eta\rangle = 0 \qquad h \in H \quad \eta \in H^*. \tag{2.28}$$

The expression (2.28) can be rewritten as

$$\langle h, \kappa\lambda\rangle = \langle \Delta(h), \kappa \otimes \lambda\rangle = \sum_{i\in I} \langle a_i(h), \kappa\rangle_1 \langle l_i, \lambda\rangle_2 = \left\langle \sum_{i\in I} \langle a_i(h), \kappa\rangle_1 l_i, \lambda \right\rangle_2 = 0$$

where $\kappa \in K^*, \lambda \in L^*$. Since $\langle\cdot,\cdot\rangle_2$ is non-degenerate and $\langle\cdot,\cdot\rangle_1$ is also non-degenerate, we find that $a_i(h) = 0$, hence $\Delta(h) = 0$. Finally, using the counit axiom

$$h = (\epsilon \otimes \mathrm{id}) \circ \Delta(h) = 0 \tag{2.29}$$

we have demonstrated that the pairing is left non-degenerate. In a similar way, it is proven that the pairing is non-degenerate at the right. Using the fact that the last equality of (2.3) is a consequence of the two first ones when the pairing is non-degenerate, we conclude that the bilinear form (2.23), which is a bialgebra pairing, is also a pairing of Hopf algebras. □

Corollary 2.1. *With the pairing and the notation defined in the previous theorem if* (k_m) *and* (κ_m) *are dual bases for K and* K^**, and* (l_n) *and* (λ_n) *are dual bases for L and* L^**, then* $(k_m l_n)$ *and* $(\kappa^m \lambda^n)$ *are dual bases for H and* H^**. In other words, if* $\langle k_m, \kappa^{m'}\rangle = \delta_m^{m'}$ *and* $\langle l_n, \lambda^{n'}\rangle = \delta_n^{n'}$ *then* $\langle k_m l_n, \kappa^{m'}\lambda^{n'}\rangle = \delta_m^{m'}\delta_n^{n'}$.

In the case of left–right bicrossproduct there is a similar result.

Theorem 2.2. *Let us consider the bicrossproduct Hopf algebra* $H = K \triangleright\!\!\blacktriangleleft L$*. Supposing that K and L are equipped with* $*$*-structures with the following compatibility relation*

$$(l \triangleleft k)^* = l^* \triangleleft S(k)^*. \tag{2.30}$$

Then the expression

$$(kl)^* = l^* k^* \qquad k \in K \quad l \in L \tag{2.31}$$

determines a ∗-structure on the algebra sector of H.

Proof. The definition of a ∗-structure on H has to be consistent with the fact that the algebra structure is an antimorphism, i.e.,

$$(lk)^* = k^* l^* \qquad k \in K \quad l \in L. \tag{2.32}$$

Since the product on H establishes that

$$lk = k_{(1)}(l \triangleleft k_{(2)}) \tag{2.33}$$

and according to the definition (2.31)

$$(lk)^* = (l \triangleleft k_{(2)})^* (k_{(1)})^*. \tag{2.34}$$

Using the product on H we obtain

$$(lk)^* = [(k_{(1)})^*]_{(1)} \{(l \triangleleft k_{(2)})^* \triangleleft [(k_{(1)})^*]_{(2)}\}. \tag{2.35}$$

Taking into account (2.30) and that the ∗-structure on K is a coalgebra morphism the equality (2.35) becomes

$$(lk)^* = (k_{(1)})^* \{l^* \triangleleft [S(k_{(3)})^* k^*_{(2)}]\}. \tag{2.36}$$

Finally, the property characterizing the antipode reduces this expression to (2.32). □

3. Induced representations for quantum algebras

The algebras involved in this work are equipped with a bicrossproduct structure, hence we work with several actions. In order to avoid any confusion we use the following notation (or the symmetric symbols for the corresponding right actions and coactions):

$\triangleright$(◀): actions (coactions) of the bicrossproduct structure;

$\vdash$: induced and inducting representations;

$\succ$: regular actions.

In the following, we show that the problem of the determination of the induced representations is reduced to the expression of products in normal ordering. The next result will be very useful for this purpose.

Proposition 3.1. *Let A be an associative algebra. The following relations hold*

$$\begin{aligned} a^m a' &= \sum_{k=0}^{m} \binom{m}{k} \mathrm{ad}^{\mathrm{l}\,k}_a(a') a^{m-k} \\ a' a^m &= \sum_{k=0}^{m} \binom{m}{k} a^{m-k} \mathrm{ad}^{\mathrm{r}\,k}_a(a') \end{aligned} \qquad \forall a, a' \in A \quad m \in \mathbb{N} \tag{3.1}$$

where

$$\mathrm{ad}^{\mathrm{l}}_a(a') = aa' - a'a = [a, a'] \qquad \mathrm{ad}^{\mathrm{r}}_a(a') = a'a - aa' = [a', a]. \tag{3.2}$$

Proof. The demonstration is by induction. The relations (3.1) are trivial identities for $m = 0$. Let us suppose that the first expression is true for $m \in \mathbb{N}$, then for $m + 1$ we have

$$\begin{aligned} a^{m+1}a' = a(a^m a') &= a \sum_{k=0}^{m} \binom{m}{k} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m-k} \\ &= \sum_{k=0}^{m} \binom{m}{k} \left[\mathrm{ad}_a^{\mathrm{l}\,k}(a')a + \mathrm{ad}_a^{\mathrm{l}\,k+1}(a')\right]a^{m-k} \\ &= \sum_{k=0}^{m} \binom{m}{k} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m-k+1} + \sum_{k=0}^{m} \binom{m}{k} \mathrm{ad}_a^{\mathrm{l}\,k+1}(a')a^{m-k} \\ &= \sum_{k=0}^{m} \binom{m}{k} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m-k+1} + \sum_{k=1}^{m+1} \binom{m}{k-1} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m-k+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m+1-k}. \end{aligned} \tag{3.3}$$

The proof of the second identity (3.1b) is similar. □

Note that in an appropriate topological context, where the convergence and the reordering of series are permitted, expressions (3.1) lead to the usual relation between adjoint action and exponential mapping:

$$\begin{aligned} e^a a' &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m-k} \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m-k} \\ &= \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \frac{1}{k!(m-k)!} \mathrm{ad}_a^{\mathrm{l}\,k}(a')a^{m-k} \\ &= e^{\mathrm{ad}_a^{\mathrm{l}}}(a')e^a \end{aligned} \tag{3.4}$$

or equivalently

$$e^a a' e^{-a} = e^{\mathrm{ad}_a^{\mathrm{l}}}(a'). \tag{3.5}$$

Taking into account that $\mathrm{ad}_{-a}^{\mathrm{l}} = \mathrm{ad}_a^{\mathrm{r}}$ we obtain an analogous relation for the other adjoint action

$$e^{-a} a' e^{a} = e^{\mathrm{ad}_a^{\mathrm{r}}}(a'). \tag{3.6}$$

3.1. General case

Let us consider a non-degenerate triplet $(H, \mathcal{H}, \langle\cdot,\cdot\rangle)$. Let L be a commutative subalgebra of H and $\{l_1, \dots, l_s\}$, a system of generators of L which can be completed with $\{k_1, \dots, k_r\}$, in such a way that $(l_n)_{n\in\mathbb{N}^s}$ is a basis of L and $(k_m l_n)_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s}$ a basis of H. Moreover, suppose that there is a system of generators in $\mathcal{H}$, $\{\kappa_1, \dots, \kappa_r, \lambda_1, \dots, \lambda_s\}$, such that $(\kappa^m\lambda^n)_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s}$ is a basis of $\mathcal{H}$ dual to that of H with the pairing

$$\langle k_m l_n, \kappa^{m'}\lambda^{n'}\rangle = m!n!\,\delta_m^{m'}\delta_n^{n'}. \tag{3.7}$$

We are interested in the description of the representation induced by the character of L determined by $a = (a_1, \ldots, a_s) \in \mathbb{K}^s$, i.e.,

$$1 \dashv l_n = a_n = a_1^{n_1} \cdots a_s^{n_s} \qquad n \in \mathbb{N}^s. \tag{3.8}$$

The elements f of $\mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ verifying the invariance condition

$$f(hl) = f(h) \dashv l \qquad \forall l \in L \quad \forall h \in H \tag{3.9}$$

constitute the carrier space $\mathbb{K}^{\uparrow} = \mathrm{Hom}_L(H, \mathbb{K})$ of the induced representation. Identifying $\mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ with $\mathcal{H}$ using the pairing, the elements of $f \in \mathbb{K}^{\uparrow}$ can be written as

$$f = \sum_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s} f_{mn} \kappa^m \lambda^n. \tag{3.10}$$

The equivariance condition (3.9)

$$\langle hl, f\rangle = \langle h, f\rangle \dashv l \qquad \forall l \in L \quad \forall h \in H \tag{3.11}$$

combined with duality gives the following relation between the coefficients f_{mn}

$$m!n! f_{mn} = \langle k_m l_n, f\rangle = \langle k_m, f\rangle a_n = m! f_{m0} a_n. \tag{3.12}$$

Hence, the elements of the carrier space of the induced representation are

$$f = \kappa\psi \qquad \kappa \in \mathcal{K} \tag{3.13}$$

where $\psi = e^{a_1\lambda_1} \cdots e^{a_s\lambda_s}$, and $\mathcal{K}$ is the subspace of $\mathcal{H}$ generated by the linear combinations of the ordered monomials $(\kappa^m)_{m\in\mathbb{N}^r}$. Since ψ is invertible (it is product of exponentials) there is an isomorphism between the vector spaces $\mathcal{K}$ and $\mathbb{K}^{\uparrow}$ given by $\kappa \to \kappa\psi$.

The action of $h \in H$ over the elements of $\mathbb{K}^{\uparrow}$ is determined after knowing the action over the basis elements $(\kappa^p\psi)_{p\in\mathbb{N}^r}$ of this space. So, putting

$$(\kappa^p\psi) \dashv h = \sum_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s} [h]^p_{mn} \kappa^m \lambda^n \qquad p \in \mathbb{N}^r \tag{3.14}$$

the constants $[h]^p_{mn}$ can be evaluated by means of duality

$$m!n![h]^p_{mn} = \langle(\kappa^p\psi) \dashv h, k_m l_n\rangle = \langle\kappa^p\psi, hk_m l_n\rangle = \langle\kappa^p\psi, hk_m\rangle a_n. \tag{3.15}$$

Note that, due to the properties of the action, we only need to compute the action for the generators of H. Finally, we see that the whole problem is reduced to writing the product hk_m in normal ordering to obtain the value of the paring in equation (3.15). In many cases this task is very cumbersome. Our objective is to prove that we can take advantage of the bicrossproduct structure in order to simplify the computations.

3.2. Quantum algebras with bicrossproduct structure

In the following, we restrict ourselves to Hopf algebras having a bicrossproduct structure like $H = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$, such that the first factor is cocommutative and the second commutative.

We are interested in the construction of the representations induced by 'real' characters of the commutative sector $\mathcal{L}$. We show that the solution of this problem can be reduced to the study of certain dynamical systems which present, in general, a nonlinear action.

Let us start by adapting the construction presented in the previous subsection 3.1 to the bicrossproduct Hopf algebras $H = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$. Let us suppose that the algebras $\mathcal{K}$ and $\mathcal{L}$ are finite generated by the sets $\{k_i\}_{i=1}^r$ and $\{l_i\}_{i=1}^s$, respectively, such that the generators k_i are primitive.

We also assume that $(k_n)_{n\in\mathbb{N}^r}$ and $(l_m)_{m\in\mathbb{N}^s}$ are bases of the underlying vector spaces to $\mathcal{K}$ and $\mathcal{L}$, respectively. Let $\mathcal{K}^*$ and $\mathcal{L}^*$ be the dual algebras of $\mathcal{K}$ and $\mathcal{L}$, respectively, having dual systems to those of $\mathcal{K}$ and $\mathcal{L}$ with analogue properties to them. Hence, duality between H and H^* is given by

$$\langle k_m l_n, \kappa^{m'} \lambda^{n'} \rangle = m!n!\, \delta_m^{m'} \delta_n^{n'}. \tag{3.16}$$

As we will see later these hypotheses are not, in fact, too restrictive. All these generator systems will be used to described the induced representations.

Let us consider the character of $\mathcal{L}$ labelled by $a \in \mathbb{C}^s$

$$1 \dashv l_n = a_n \qquad n \in \mathbb{N}^s. \tag{3.17}$$

The discussion of subsection 3.1 allows us to state the following theorem:

Theorem 3.1. *The carrier space, $\mathbb{C}^\uparrow$, of the representation of H induced by the character a of $\mathcal{L}$ (see equation (3.17)) is isomorphic to $\mathcal{K}^*$ and is constituted by the elements of the form $\kappa\psi$, where $\kappa \in \mathcal{K}^*$ and*

$$\psi = \mathrm{e}^{a_1\lambda_1}\, \mathrm{e}^{a_2\lambda_2} \cdots \mathrm{e}^{a_s\lambda_s}. \tag{3.18}$$

The induced action is given by

$$f \dashv h = \sum_{m\in\mathbb{N}^r} \kappa^m \left\langle h\frac{k_m}{m!}, f \right\rangle \psi \qquad h \in H \quad f \in \mathbb{C}^\uparrow. \tag{3.19}$$

The action of the generators of $\mathcal{K}$ and $\mathcal{L}$ in the induced representation will be given in the next theorem, which needs the introduction of some new concepts.

Since $\mathcal{L}$ is commutative, it can be identified with the algebra of functions $F(\mathbb{R}^s)$ by means of the algebra morphism $\mathcal{L} \to F(\mathbb{R}^s)$, $(l \mapsto \tilde{l})$, which maps the generators of $\mathcal{L}$ into the canonical projections

$$\tilde{l}_j(x) = x_j \qquad \forall x = (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s \quad j = 1, 2, \ldots, s. \tag{3.20}$$

The $*$-structure, keeping the generators chosen in $\mathcal{L}$ invariant, is distinguished in a natural way by the above identification

$$l_j^* = l_j \qquad j = 1, 2, \ldots, s. \tag{3.21}$$

The characters (3.17) compatible with (3.21) are ‘real’, i.e., determined by the elements $a \in \mathbb{R}^n \subset \mathbb{C}^n$. Henceforth we restrict ourselves to them. They can be written now as

$$1 \dashv l = \tilde{l}(a) \qquad a \in \mathbb{R}^n. \tag{3.22}$$

The right action of $\mathcal{K}$ on $\mathcal{L}$ can be translated to $F(\mathbb{R}^s)$ because the generators of $\mathcal{K}$ are primitive and, hence, they act by derivations on the $\mathcal{K}$-module algebra of $\mathcal{K} \blacktriangleright\!\!\!\triangleleft \mathcal{L}$. Thus, the generators k_i induce vector fields, X_i, on $\mathbb{R}^s$ determined by

$$X_i\tilde{l} = \widetilde{l \triangleleft k_i} \qquad i = 1, 2, \ldots, r. \tag{3.23}$$

The corresponding flow, $\Phi_i : \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^s$, is implicitly defined by

$$(X_i f)(x) = (Df_{x,\Phi_i})(0) \tag{3.24}$$

where $f_{x,\Phi_i}(t) = f \circ \Phi_i^t(x)$ and D is the derivative operator over real variable functions. Notice that, in general, the one-parameter group of transformations associated to the flow Φ_i is not globally defined.

Proposition 3.2. *In the Hopf algebra $H = \mathcal{K} \blacktriangleright\!\!\!\triangleleft \mathcal{L}$ the following relation holds*

$$l k_m = \sum_{p\leqslant m} \binom{m}{p} k_{m-p}(l \triangleleft k_p) \qquad \forall l \in \mathcal{L} \quad \forall m \in \mathbb{N}^r \tag{3.25}$$

where the multi-combinatorial number is defined as the product of usual combinatorial numbers or through multi-factorials

$$\binom{m}{p} = \prod_{i=1}^{r} \binom{m_i}{p_i} = \frac{m!}{p!(m-p)!} \tag{3.26}$$

where the ordered relation over the multi-indices is given by

$$p \leqslant m \quad \Leftrightarrow \quad p_1 \leqslant m_1, p_2 \leqslant m_2, \ldots, p_r \leqslant m_r \tag{3.27}$$

and if $p \leqslant m$ *the difference between m and p is well defined in* $\mathbb{N}^r$ *by*

$$m - p = (m_1 - p_1, m_2 - p_2, \ldots, m_r - p_r). \tag{3.28}$$

Proof. Let us consider an element l of $\mathcal{L}$ and a generator k_i of $\mathcal{K}$ in the associative algebra $\mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$. Taking into account the definition of the product in $\mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$ and that the generators k_i are primitive, we can write

$$\mathrm{ad}^{\mathrm{r}}_{k_i}(l) = [l, k_i] = l \triangleleft k_i \qquad \left(\mathrm{ad}^{\mathrm{r}}_{k_i}\right)^p (l) = l \triangleleft k_i^p. \tag{3.29}$$

From the second formula of (3.1) after identifying $a' = l$ and $a = k_i$ we obtain

$$l k_i^m = \sum_{p \leqslant m} \binom{m}{p} k_i^{m-p} \big(l \triangleleft k_i^p\big). \tag{3.30}$$

This formula is valid for $m \in \mathbb{N}$. The validity of the expression for a multi-index $m \in \mathbb{N}^r$ is a direct consequence of the properties of the action $\triangleleft$ and of the definitions of the multi-objects that have been introduced. □

Theorem 3.2. *The explicit action of the generators of* $\mathcal{K}$ *and* $\mathcal{L}$ *in the induced representation of theorem 3.1 realized in the space* $\mathcal{K}^*$ *is given by the following expressions*

$$\kappa \dashv k_i = \kappa \prec k_i \qquad \kappa \dashv l_j = \kappa \hat{l}_j \circ \Phi_{(\kappa_1, \kappa_2, \ldots, \kappa_r)}(a) \tag{3.31}$$

where $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s\}$, *the symbol* $\prec$ *denotes the regular action of* $\mathcal{K}$ *on* $\mathcal{K}^*$, *and* $\Phi_{(\kappa_1, \kappa_2, \ldots, \kappa_r)} = \Phi_r^{\kappa_r} \circ \cdots \circ \Phi_2^{\kappa_2} \circ \Phi_1^{\kappa_1}$.

Proof. For the first expression we apply (3.19) to the case $h = k_r$

$$\begin{aligned}(\kappa\psi) \dashv k_i &= \sum_{m \in \mathbb{N}^r} \kappa^m \left\langle k_i \frac{k_m}{m!}, \kappa\psi \right\rangle \psi \\ &= \sum_{m \in \mathbb{N}^r} \kappa^m \left\langle k_i \frac{k_m}{m!}, \kappa \right\rangle \langle 1_{\mathcal{L}}, \psi \rangle \psi \\ &= \sum_{m \in \mathbb{N}^r} \kappa^m \left\langle \frac{k_m}{m!}, \kappa \prec k_i \right\rangle \psi \\ &= (\kappa \prec k_i)\psi. \end{aligned} \tag{3.32}$$

We use that $\langle 1_{\mathcal{L}}, \psi \rangle = 1$ in the third equality, and the last is based on the fact that $\frac{1}{m!}\kappa^m \otimes k_m$ is the T-matrix [29] of the pair $(\mathcal{K}^*, \mathcal{K})$.

The computation of the action of l_j is more complicated:

$$\begin{aligned}(\kappa\psi) \dashv l_j &= \sum_{m \in \mathbb{N}^r} \kappa^m \left\langle l_j \frac{\kappa_m}{m!}, \kappa\psi \right\rangle \psi \\ &= \sum_{m \in \mathbb{N}^r} \kappa^m \left\langle \sum_{p \leqslant m} \binom{m}{p} \frac{1}{m!} k_{m-p} (l_j \triangleleft k_p), \kappa\psi \right\rangle \psi \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\in\mathbb{N}^r}\sum_{p\leqslant m}\frac{1}{p!(m-p)!}\kappa^m\langle k_{m-p}(l_j\triangleleft k_p),\kappa\psi\rangle\psi \\
&= \sum_{p\in\mathbb{N}^r}\sum_{m\in p+\mathbb{N}^r}\frac{1}{p!(m-p)!}\kappa^m\langle k_{m-p}(l_j\triangleleft k_p),\kappa\psi\rangle\psi \\
&= \sum_{p\in\mathbb{N}^r}\sum_{m\in\mathbb{N}^r}\frac{1}{p!m!}\kappa^{m+p}\langle k_m(l_j\triangleleft k_p),\kappa\psi\rangle\psi \\
&= \sum_{m\in\mathbb{N}^r}\frac{1}{m!}\kappa^m\langle k_m,\kappa\psi\rangle\sum_{p\in\mathbb{N}^r}\frac{1}{p!}\kappa^p(1\dashv(l_j\triangleleft k_p))\psi \\
&= \sum_{m\in\mathbb{N}^r}\frac{1}{m!}\kappa^m\langle k_m,\kappa\rangle\sum_{p\in\mathbb{N}^r}\frac{1}{p!}\kappa^p(1\dashv(l_j\triangleleft k_p))\psi \\
&= \kappa\sum_{p\in\mathbb{N}^r}\frac{1}{p!}\kappa^p(1\dashv(l_j\triangleleft k_p))\psi \\
&= \kappa\sum_{p\in\mathbb{N}^r}\frac{1}{p!}\kappa^p\widehat{l_j\triangleleft k_p}(a)\psi \\
&= \kappa\left[\sum_{p\in\mathbb{N}^r}\frac{1}{p!}\kappa^pX_p\bigg|_a\hat{l}_j\right]\psi.
\end{aligned}
\tag{3.33}
$$

In the second equality of (3.33), proposition 3.2 has been used. The next three are simple reorderings of the sums. The sixth equality is a consequence of the equivariance property and of the commutativity in $\mathcal{K}^*$. The definitions of the duality form in the bicrossproduct structure, of the T-matrix of the algebra $\mathcal{K}$ and of the identification of L with the algebra of functions $F(\mathbb{R}^s)$ are successively applied in the next equalities. Finally, we define $X_p = X_r^{p_r}\dots X_2^{p_2}X_1^{p_1}$ in terms of the vector fields associated with the generators k_i.

On the other hand, from relation (3.24) between the flow Φ_i and X_i we obtain

$$
f\circ\Phi_i^t(x) = f_{x,\Phi_i}(t) = (e^{tD}f_{x,\Phi_i})(0) = \sum_{n=0}^{\infty}\frac{1}{n!}t^n(D^nf_{x,\Phi_i})(0) = \sum_{n=0}^{\infty}\frac{1}{n!}t^n\left(X_i^nf\right)(x)
\tag{3.34}
$$

for any regular function $f\in F(\mathbb{R}^s)$. So, to obtain the expression of the action established in this theorem it suffices to take $f=\hat{l}_j$, $x=a$, to replace formally the real number t by κ^i, to make successively $i=1,\dots,r$ and, finally, to substitute the obtained relation in (3.33). □

We remark that the inverse order in product $X_p = X_r^{p_r}\dots X_2^{p_2}X_1^{p_1}$ and in the flow composition $\Phi_{t_1,t_2,\dots,t_r} = \Phi_{t_r}\dots\Phi_{t_2}\Phi_{t_1}$ is due to the right nature of the action of $\mathcal{K}$ on $\mathcal{L}$.

When H is the deformed enveloping algebra of a semi-direct product with an Abelian kernel and the sector $\mathcal{K}$ is nondeformed, then the first expression of (3.31) means that the representation of $\mathcal{K}$ is the same that in the nondeformed case. On the other hand, the generators of $\mathcal{L}$ act as multiplication operators affected by the deformation.

4. Modules and representations

The close relationship between representations and modules (see [1]) allow us to reformulate the theory of induced representations for quantum algebras that we have developed in the previous section from the perspective of module theory.

4.1. Regular modules

The objective of this section is to describe the four regular H-modules associated to a Hopf algebra H: $(H, \prec, H)$, $(H^*, \succ, H)$, $(H, \succ, H)$ and $(H^*, \prec, H)$; H^* is the dual of H in the sense of non-degenerate pairing (see subsection 2.1).

It is well known that theorems exist which prove that, essentially, all the commutative or cocommutative Hopf algebras are of the form $F(G)$ or $\mathbb{K}[G]$ (or $U(\mathfrak{g})$) for some group G [28]. So, the type of bicrossproduct that we consider can be described as

$$H = \mathbb{C}[K] \triangleright\!\blacktriangleleft F(L) \qquad \text{or} \qquad H = U(\mathfrak{k}) \triangleright\!\blacktriangleleft F(L) \tag{4.1}$$

where K and L are finite groups or Lie groups.

We focus our attention on the case that both K and L are Lie groups with associated Lie algebras $\mathfrak{k}$ and $\mathfrak{l}$, respectively. In this way, the dual of H will be

$$H^* = F(K) \blacktriangleright\!\triangleleft U(\mathfrak{l}). \tag{4.2}$$

The key point for an effective description of the regular modules is to use elements of H and H^* that can be decomposed as follows

$$k\lambda \in H \qquad k \in K \qquad \lambda \in F(L) \qquad \kappa l \in H^* \qquad \kappa \in F(K) \qquad l \in L. \tag{4.3}$$

We will see that these elements describe completely the structures of the regular H-modules and are more suitable than the bases of ordered monomials.

Theorem 4.1. *Let us consider elements $k, k' \in K, \lambda, \lambda' \in F(L), \kappa \in F(K)$ and $l \in L$. The action on each of the four regular H-modules is:*

$$\begin{aligned}
&(H, \prec, H)\colon (k\lambda) \prec k' = kk'(\lambda \triangleleft k') && (k\lambda) \prec \lambda' = k\lambda\lambda' \\
&(H^*, \succ, H)\colon k' \succ (\kappa l) = (k' \succ \kappa)(k' \triangleright l) && \lambda' \succ (\kappa l) = \kappa(\lambda' \succ l) \\
&(H, \succ, H)\colon k' \succ (k\lambda) = k'k\lambda && \lambda' \succ (k\lambda) = k(\lambda' \triangleleft k)\lambda \\
&(H^*, \prec, H)\colon (\kappa l) \prec k' = (\kappa \prec k')l && (\kappa l) \prec \lambda' = \kappa\langle l^{(1)}, \lambda'\rangle l^{(2)} l.
\end{aligned} \tag{4.4}$$

Proof.

(1) The results relative to the modules $(H, \prec, H)$ and $(H, \succ, H)$ only require the use of the product defined on the semi-direct product of algebras $U(\mathfrak{k}) \triangleright\!< F(L)$ (remember that for arbitrary elements $k, k' \in U(\mathfrak{k})$ and $\lambda, \lambda' \in F(L)$ such a product is given by $(k \otimes \lambda)(k' \otimes \lambda') = kk'_{(1)} \otimes (\lambda \triangleleft k'_{(2)})\lambda'$). In order to evaluate the action of k' we take into account that $\Delta(k') = k' \otimes k'$.

(2) In the module algebra $(H^*, \succ, H)$ the action of k' is obtained by

$$\begin{aligned}
\langle k\lambda, k' \succ (\kappa l)\rangle &= \langle (k\lambda) \prec k', \kappa l\rangle = \langle kk'(\lambda \triangleleft k'), \kappa l\rangle \\
&= \langle kk', \kappa\rangle\langle \lambda \triangleleft k', l\rangle = \langle k, k' \succ \kappa\rangle\langle \lambda, k' \triangleright l\rangle \\
&= \langle k\lambda, (k' \succ \kappa)(k' \triangleright l)\rangle.
\end{aligned} \tag{4.5}$$

The action of l' is obtained in an analogous way

$$\begin{aligned}
\langle k\lambda, \lambda' \succ (\kappa l)\rangle &= \langle k\lambda \prec \lambda', \kappa l\rangle = \langle k\lambda\lambda', \kappa l\rangle \\
&= \langle k, \kappa\rangle\langle \lambda\lambda', l\rangle = \langle k, \kappa\rangle\langle \lambda, \lambda' \succ l\rangle \\
&= \langle k\lambda, \kappa(\lambda' \succ l)\rangle.
\end{aligned} \tag{4.6}$$

Notice that in the first expression of equations (4.5) and (4.6) the symbol $\succ$ represents the regular action of $(H^*, \succ, H)$, but in the last expression it denotes the action of $(F(K), \succ, U(\mathfrak{g}))$ and of $(F(L), \succ, U(\mathfrak{l}))$, respectively.

(3) When the regular module $(H^*, \prec, H)$ is taken into consideration, the following chains of equalities determine the action of k' and λ', respectively:

$$\begin{aligned}\langle(\kappa l) \prec k', k\lambda\rangle &= \langle \kappa l, k' \succ (k\lambda)\rangle = \langle \kappa l, k'k\lambda\rangle \\ &= \langle \kappa, k'k\rangle\langle l, \lambda\rangle = \langle \kappa \prec k', k\rangle\langle l, \lambda\rangle = \langle(\kappa \prec k')l, k\lambda\rangle \end{aligned} \quad (4.7)$$

$$\begin{aligned}\langle(\kappa l) \prec \lambda', k\lambda\rangle &= \langle \kappa l, \lambda' \succ (k\lambda)\rangle = \langle \kappa l, k(\lambda' \triangleleft k)\lambda\rangle \\ &= \langle \kappa, k\rangle\langle l, (\lambda' \triangleleft k)\lambda\rangle = \langle \kappa, k\rangle\langle l, (\lambda' \triangleleft k)\rangle\langle l, \lambda\rangle \\ &= \langle \kappa, k\rangle\langle l^{(1)}, \lambda'\rangle\langle l^{(2)}, k\rangle\langle l, \lambda\rangle = \langle \kappa l^{(2)}, k\rangle\langle l^{(1)}, \lambda'\rangle\langle l, \lambda\rangle \\ &= \langle \kappa l^{(2)} l, k\lambda\rangle\langle l^{(1)}, \lambda'\rangle = \langle \kappa\langle l^{(1)}, \lambda'\rangle l^{(2)} l, k\lambda\rangle. \end{aligned} \quad (4.8)$$

□

Note that: (i) the action (4.8) is described in terms of the structure of $U(\mathfrak{l})$ as a right $F(K)$-comodule; and (ii) except for the term $\langle l^{(1)}, \lambda'\rangle l^{(2)}$ including a coaction, the action on the regular modules appears to be described by means of other actions, most of them regular.

From a computational point of view, the following proposition and its corollary are very useful, since they reduce the description of the regular modules to the study of the action of K on L associated with the structure of the $U(\mathfrak{k})$-module of $F(L)$.

Let us start by fixing the notation to be used. Let r and s be the dimensions of the groups K and L, respectively. Let us consider the basis $(k_i)_{i=1}^r$ of $\mathfrak{k}$ and $(l_j)_{j=1}^s$ of $\mathfrak{l}$, and the local coordinate systems of the second kind associated to the above bases $(\kappa_i)_{i=1}^r$ and $(\lambda_j)_{j=1}^s$. Remember that using multi-index notation we have

$$\langle k_n, \kappa^{n'}\rangle = n!\,\delta_n^{n'} \qquad \langle l_m, \lambda^{m'}\rangle = m!\,\delta_m^{m'} \qquad n, n' \in \mathbb{N}^r \quad m, m' \in \mathbb{N}^s. \quad (4.9)$$

Finally, let us denote by k the inverse map of the coordinate system (κ_i), i.e.,

$$\begin{aligned} k : \mathbb{R}^r &\longrightarrow K \\ t &\mapsto e^{t_1 k_1} e^{t_2 k_2} \cdots e^{t_r k_r}. \end{aligned} \quad (4.10)$$

Proposition 4.1. *For every $\lambda \in F(L)$ and $l \in L$ the following relation holds*

$$\langle l^{(1)}, \lambda\rangle l^{(2)} = \sum_{n\in\mathbb{N}^r} \frac{1}{n!}\langle k_n \triangleright l, \lambda\rangle \kappa^n = \lambda(k(\kappa) \triangleright l). \quad (4.11)$$

Proof. Let us rewrite the coaction at the right of $F(K)$ on $l \in L$ as

$$\blacktriangleright l \equiv l^{(1)} \otimes l^{(2)} = \sum_{(m,n)\in\mathbb{N}^s\times\mathbb{N}^r} [l]_n^m l_m \otimes \kappa^n. \quad (4.12)$$

The pairing defined in the bicrossproduct in accordance with theorem 2.1 gives the coordinates of $\blacktriangleright l$ in terms of the action (dual of the coaction) of $U(\mathfrak{k})$ on $U(\mathfrak{l})$:

$$[l]_m^n = \frac{1}{m!n!}\langle \lambda^m \otimes k_n, \blacktriangleright l\rangle = \frac{1}{m!n!}\langle \lambda^m, k_n \triangleright l\rangle. \quad (4.13)$$

Inserting the last expression in equation (4.12) we easily obtain

$$\begin{aligned}\langle l^{(1)}, \lambda\rangle l^{(2)} &= \sum_{(m,n)\in\mathbb{N}^s\times\mathbb{N}^r} \frac{1}{m!n!}\langle \lambda^m, k_n \triangleright l\rangle\langle l_m, \lambda\rangle\kappa^n \\ &= \sum_{(m,n)\in\mathbb{N}^s\times\mathbb{N}^r} \frac{1}{m!n!}\langle k_n \triangleright l, \lambda^m\rangle\langle l_m, \lambda\rangle\kappa^n. \end{aligned} \quad (4.14)$$

The sum on m takes account of the action of the T-matrix associated with the pair $(U(\mathfrak{l}), F(L))$ and, hence, the expression (4.14) is simplified:

$$\langle l^{(1)}, \lambda \rangle l^{(2)} = \sum_{n \in \mathbb{N}^r} \frac{1}{n!} \langle k_n \triangleright l, \lambda \rangle \kappa^n. \tag{4.15}$$

On the other hand, since

$$\lambda(k(t) \triangleright l) = \langle k(t) \triangleright l, \lambda \rangle = \sum_{n \in \mathbb{N}^r} \frac{1}{n!} \langle k_n \triangleright l, \lambda \rangle t^n \tag{4.16}$$

in order to obtain $\langle l^{(1)}, \lambda \rangle l^{(2)}$ it is enough to perform the formal substitution $t_i \to \kappa_i$ in the expression $\lambda(k(t) \triangleright l)$. □

The above proposition gives an expression for $\langle l^{(1)}, \lambda \rangle l^{(2)}$ completely independent of the bases chosen in the algebras.

Corollary 4.1. *Let $\hat{l}$ be the map*

$$\begin{array}{llll} \hat{l} : K & \longrightarrow & L \\ k & \mapsto & k \triangleright l \end{array} \tag{4.17}$$

projecting the group K on the orbit passing through $l \in L$. Then, for any $\lambda \in F(L)$ and any $l \in L$ we have

$$\langle l^{(1)}, \lambda \rangle l^{(2)} = \lambda \circ \hat{l}. \tag{4.18}$$

Taking into account that in $(U(\mathfrak{l}), \succ, F(L))$ it is verified that

$$\lambda \succ l = \lambda(l) l \qquad \forall \lambda \in F(L) \quad \forall l \in L. \tag{4.19}$$

Theorem 4.1 can be rewritten in a more explicit way.

Theorem 4.2. *The action on each of the four regular H-modules is*

$$\begin{array}{lll} (H, \prec, H)\colon (k\lambda) \prec k' = kk'(\lambda \triangleleft k') & (k\lambda) \prec \lambda' = k\lambda\lambda' \\ (H^*, \succ, H)\colon k' \succ (\kappa l) = (k' \succ \kappa)(k' \triangleright l) & \lambda' \succ (\kappa l) = \lambda'(l)\kappa l \\ (H, \succ, H)\colon k' \succ (k\lambda) = k'k\lambda & \lambda' \succ (k\lambda) = k(\lambda' \triangleleft k)\lambda \\ (H^*, \prec, H)\colon (\kappa l) \prec k' = (\kappa \prec k')l & (\kappa l) \prec \lambda' = \kappa(\lambda' \circ \hat{l})l \end{array} \tag{4.20}$$

where $k, k' \in K$, $\lambda, \lambda' \in F(L)$, $\kappa \in F(K)$ and $l \in L$.

This theorem does not make reference to the nature of Lie groups K and L, since it is formulated in terms of the regular actions and those associated to the bicrossproduct structure. Thus, the theorem may be applied to other kinds of groups.

Note that, in general, the action of K on L is not globally defined. Hence, $\hat{l}$ (4.17) only projects, in reality, a neighbourhood of the identity into the orbit of l. Henceforth, $\lambda \circ \hat{l}$ does not define, in general, a map over the whole K and the expression $(\kappa l) \prec \lambda' = \kappa(\lambda' \circ \hat{l})l$ only has sense enlarging the space $F(K)$, for instance, including it inside spaces of formal series.

In conclusion, we can say that in the description of the regular actions the computation of the left action of the group K on the group L is really the most important fact. From this point of view, the deformations used in this work may be interpreted as one-parameter families of nonlinear actions homotopically equivalent to the linear actions of the nondeformed cases.

4.2. Co-spaces and induction

In the context of noncommutative geometry the manifold X is replaced by the algebra $F(X)$ of $\mathcal{C}^\infty$ $\mathbb{C}$-valued functions on X as well as the Lie group G by the enveloping algebra $U(\mathfrak{g})$ of its Lie algebra $\mathfrak{g}$. Since $(F(X), \triangleright, U(\mathfrak{g}))$ is a module algebra over the Hopf algebra $U(\mathfrak{g})$, we can generalize the concept of G-space in algebraic terms [16].

Let H be a Hopf algebra. A left (right) H-co-space is a module algebra $(A, \triangleright, H)$ $((A, \triangleleft, H))$.

The morphisms among H-co-spaces are the morphisms of H-modules and the concepts of subco-space or quotient co-space are equivalent to module subalgebra or quotient module algebra, respectively. We have adopted the term of *co-space* instead of *space* to stress the dual character of A as a way of describing the initial geometric object.

Given a pair of algebras with a non-degenerate pairing $(H, H', \langle\cdot,\cdot\rangle)$, we obtain, via dualization of the regular actions, the regular H-co-spaces $(H', \succ, H)$ and $(H', \prec, H)$.

The explicit description of the four regular modules studied in the previous subsection allows us a complete analysis of the representations of the algebra $H = U(\mathfrak{k}) \triangleright\!\!\blacktriangleleft F(L)$ induced by the one-dimensional modules of the commutative sector. As we will see, the left co-space $(H^*, \succ, H)$ characterizes the carrier space of the induced representation and the right co-space $(H^*, \prec, H)$ determines the induced action of H on the carrier space.

Firstly, remember that the set of characters of the algebra $F(L)$ is its spectrum. An important theorem by Gelfand and Naimark [30] establishes the following isomorphism

$$\text{Spectrum}\, F(L) \simeq L. \tag{4.21}$$

Fixing $l \in L$, the character (or the corresponding right $F(L)$-module over $\mathbb{C}$) is given by

$$1 \dashv \lambda = \lambda(l) \qquad \lambda \in F(L). \tag{4.22}$$

In order to construct the representation of $H = U(\mathfrak{k}) \triangleright\!\!\blacktriangleleft F(L)$ induced by equation (4.22) let us start by determining the carrier space $\mathbb{C}^\uparrow \subset H^*$. The element $f \in H^*$ satisfies the equivariance condition if it verifies

$$\lambda \succ f = \lambda(l) f \qquad \forall \lambda \in F(L). \tag{4.23}$$

Expanding f in terms of the bases of $\mathfrak{k}$ and $\mathfrak{l}$

$$f = \sum_{(m,n)\in\mathbb{N}^r\times\mathbb{N}^s} f_m^n \kappa^m l_n \tag{4.24}$$

the equivariance condition gives the following relation among the coefficients f_m^n

$$f_m^n = \frac{1}{m!n!} f_m^0 \lambda^n(l) \qquad m \in \mathbb{N}^r \qquad n \in \mathbb{N}^s. \tag{4.25}$$

Hence, the general solution is

$$f = \left(\sum_{m\in\mathbb{N}^r} \frac{1}{m!} f_m^0 \kappa^m\right)\left(\sum_{n\in\mathbb{N}^s} \frac{1}{n!} \lambda^n(l) l_n\right) \qquad f_m^0 \in \mathbb{C}. \tag{4.26}$$

Taking into account the definition of the second-kind coordinates λ_j over the group L, the expression (4.26) can be rewritten in a more compact form

$$f = \kappa l \qquad \kappa \in F(K). \tag{4.27}$$

In other words, the carrier space of the induced representation admits a natural description in terms of products *function*/*element*, introduced in (4.3), instead of terms of monomial bases.

The right regular action describes the action on the induced module, which can be translated to $F(K)$ using the isomorphism $F(K) \to \mathbb{C}^{\uparrow}(\kappa \mapsto \kappa l)$:

$$\kappa \dashv k = \kappa \prec k \qquad \kappa \dashv \lambda = \kappa(\lambda \circ \hat{l}). \tag{4.28}$$

Comparing these expressions with those of theorem 3.2 we observe that the action of the subalgebra $U(\mathfrak{k})$ is given by the regular action. The action of the subalgebra $F(L)$ is multiplicative and the evaluation of the corresponding factor, from a computational point of view, is essentially reduced to obtain the one-parameter flows associated with the action of K on L derived from the bicrossproduct structure of the algebra H.

4.3. Equivalence and unitarity of the induced representations

Let $\dashv_l$ be the representation of $H = U(\mathfrak{k}) \triangleright\!\!\blacktriangleleft F(L)$ induced by $l \in L$ and f_k the automorphism of $F(K)$ given by the regular action of an element $k \in K$, i.e., $f_k(\kappa) = k \succ \kappa$. Since

$$[k \succ (\lambda \circ \hat{l})](k') = (\lambda \circ \hat{l})(k'k) = \lambda((k'k) \triangleright l) = [\lambda \circ \widehat{k \triangleright l}](k')$$

then we have that

$$f_k(\kappa \dashv_l \lambda) = k \succ [\kappa(\lambda \circ \hat{l})] = (k \succ \kappa)[k \succ (\lambda \circ \hat{l})] = f_k(\kappa)[\lambda \circ \widehat{k \triangleright l}] = f_k(\kappa) \dashv_{\widehat{k \triangleright l}} \lambda.$$

Finally, taking into account that the action of the subalgebra $U(\mathfrak{k})$ on the induced module is not affected by the choice of the element l in L, we conclude that the H-modules $(\mathbb{C}^{\uparrow}, \dashv_l, H)$ and $(\mathbb{C}^{\uparrow}, \dashv_{\widehat{k \triangleright l}}, H)$ are isomorphic via f_k.

The problem of the unitarity of the induced representation passes, firstly, by choosing a $*$-structure in H. The usual determination is to consider 'hermitian operators' a family of generators of H, but troubles may appear [26, 31], related with the real or complex nature of the deformation parameter. The point of view adopted here gives a simple solution of the problem: $U(\mathfrak{k})$ and $F(L)$ carry associated $*$-structures in a natural way. Explicitly,

$$k^* = k^{-1} \qquad \forall k \in K \qquad \lambda^*(l) = \overline{\lambda(l)} \qquad \forall \lambda \in F(L) \qquad \forall l \in L. \tag{4.29}$$

Choosing in H the $*$-structure associated to those given by equation (4.29), according to theorem 2.2, the problem of the unitarization is easily solved. Firstly, the action of the elements $k \in K$ shows that the space $F(K)$ has to be restricted to the square-integrable functions with respect to the right invariant Haar measure μ over K (i.e., $\mu(k \triangleright A) = \mu(A)$ with A a μ-measurable set in K). In fact, it is necessary to restrict the space $\mathcal{H} = L^2(K, \mu)$ and to consider only the space $\mathcal{H}_\infty$ of $\mathcal{C}^\infty$ functions, since the Lie algebra $U(\mathfrak{k})$ acts by means of differential operators over these functions. On the other hand, the elements of $F(L)$ act by a multiplicative factor and impose a new restriction in $\mathcal{H}_\infty$ because only the functions κ such that $\kappa(\lambda \circ \hat{l})$ is also square-integrable (supposing that the action is global in the orbit of l) will be admissible. If K is compact all that is automatically verified. In the opposite case there is a condition over the vanishing order of κ at the infinity points. The results of this discussion are summarized in the following theorem.

Theorem 4.3. *Let us consider an element $l \in L$ supporting a global action of the group K. The carrier space, $\mathbb{C}^{\uparrow}$, of the representation of H induced by the character determined by l is the set of elements of H^* of the form*

$$\kappa l \qquad \kappa \in F(K). \tag{4.30}$$

There is an isomorphism between $\mathbb{C}^{\uparrow}$ and $F(K)$ given by the map $\kappa \mapsto \kappa l$. The action induced by the elements of the form $k \in K$ and $\lambda \in F(L)$ on the space $F(K)$ is

$$\kappa \dashv k = \kappa \prec k \qquad \kappa \dashv \lambda = \kappa(\lambda \circ \hat{l}). \tag{4.31}$$

The modules induced by l and $k \triangleright l$ are isomorphic. So, the induction algorithm establishes a correspondence between the space of orbits L/K and the set of equivalence classes of representations.

If the group K is compact, the induced representation is unitary in the space, $L^2(K)$, of square-integrable functions with respect to the right invariant Haar measure, when the $*$-structure given by theorem 2.2, and applied to the natural structures of the factors of the bicrossproduct $H = U(\mathfrak{k}) \bowtie F(L)$, is considered.

4.4. Local representations

The so-called local representations [32] in the deformed version appear when we induce by representations of the subalgebra $U(\mathfrak{k})$. Let us consider the following character of $U(\mathfrak{k})$

$$\kappa \in \text{Spectrum}\, U(\mathfrak{k}) \subset F(K) \qquad k \vdash 1 = \kappa(k). \tag{4.32}$$

Since the Hopf algebra $U(\mathfrak{k})$ is, in general, non-commmutative, the set of characters may be very reduced, it may even be generated only by the counit. For this reason, an interesting problem to be investigated in the future is the study of the representations induced by representations of $U(\mathfrak{k})$ of dimension greater than one.

The carrier space of the representation induced by κ is determined by the following equivariance condition

$$f \prec k = \kappa(k) f \qquad \forall k \in U(\mathfrak{k}). \tag{4.33}$$

The algebra H^* can be considered as a left free $F(K)$-module and, hence, it is possible to fix a basis $(l_j)_{j\in J}$ of $U(\mathfrak{l})$ such that $f \in H^*$ can be expressed in a unique form as

$$f = \sum_{j\in J} \kappa_j l_j \qquad \kappa_j \in F(K). \tag{4.34}$$

The equivariance condition can be written now as

$$\sum_{j\in J} (\kappa_j \prec k) l_j = \sum_{j\in J} \kappa(k) \kappa_j l_j \qquad \forall k \in U(\mathfrak{k}). \tag{4.35}$$

Taking into account that the elements l_j constitute a basis of the $F(K)$-module, H^*, the corresponding coefficients can be equated, obtaining

$$\kappa_j \prec k = \kappa(k)\kappa_j \qquad \forall k \in U(\mathfrak{k}). \tag{4.36}$$

The previous equality (4.36) implies that

$$\kappa_j(kk') = \kappa(k)\kappa_j(k') \qquad \forall k, k' \in K. \tag{4.37}$$

Choosing k' equal to the identity element $e \in K$, we obtain

$$\kappa_j = \kappa_j(e)\kappa \qquad \forall j \in J. \tag{4.38}$$

In this way the elements of $\mathbb{C}^\uparrow$ are of the form

$$f = \kappa l \qquad l \in U(\mathfrak{l}). \tag{4.39}$$

The map $U(\mathfrak{l}) \to \mathbb{C}^\uparrow$, defined by $l \mapsto \kappa l$, is an isomorphism of vector spaces. The representation can be realized in this way on $U(\mathfrak{l})$ and the final result is

$$k \vdash l = \kappa(k) k \triangleright l \qquad \lambda \vdash l = \lambda(l) l. \tag{4.40}$$

In [19] we have applied this theory to the (1 + 1) quantum extended Galilei algebra.

5. Examples

5.1. Null-plane quantum Poincaré algebra

The null-plane quantum deformation of the (1 + 1) Poincaré algebra, $U_z(\mathfrak{p}(1,1))$, is a q-deformed Hopf algebra that in a null-plane basis, $\{P_+, P_-, K\}$, has the form [33, 34]

$$\begin{array}{lll} [K, P_+] = \frac{-1}{z}(\mathrm{e}^{-2zP_+} - 1) & [K, P_-] = -2P_- & [P_+, P_-] = 0 \\ \Delta P_+ = P_+ \otimes 1 + 1 \otimes P_+ & \Delta X = X \otimes 1 + \mathrm{e}^{-2zP_+} \otimes X & X \in \{P_-, K\} \\ \epsilon(X) = 0 & X \in \{P_\pm, K\} & \\ S(P_+) = -P_+ & S(X) = -\mathrm{e}^{2zP_+} X & X \in \{P_-, K\}. \end{array} \tag{5.1}$$

It has also a bicrossproduct structure [35]

$$U_z(\mathfrak{p}(1,1)) = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$$

where $\mathcal{K}$ is a commutative and cocommutative Hopf algebra generated by K, and $\mathcal{L}$ is the commutative Hopf subalgebra of $U_z(\mathfrak{p}(1,1))$ generated by P_+ and P_-.

The right action of $\mathcal{K}$ on $\mathcal{L}$ comes determined by

$$P_+ \triangleleft K = \frac{1}{z}(\mathrm{e}^{-2zP_+} - 1) \qquad P_- \triangleleft K = 2P_-. \tag{5.2}$$

The left coaction of $\mathcal{L}$ over the generator of $\mathcal{K}$ is

$$K \blacktriangleleft = \mathrm{e}^{-2zP_+} \otimes K. \tag{5.3}$$

In the dual Hopf algebra $F_z(P(1,1)) = \mathcal{K}^* \blacktriangleright\!\!\triangleleft \mathcal{L}^*$ let us denote by φ the generator of $\mathcal{K}^*$ and by $a_\pm$ those of $\mathcal{L}^*$. The left action of $\mathcal{L}^*$ on $\mathcal{K}^*$ is given by

$$a_+ \triangleright \varphi = 2z(\mathrm{e}^{-\varphi} - 1) \qquad a_- \triangleright \varphi = 0 \tag{5.4}$$

and the right coaction of $\mathcal{K}^*$ over the generators of $\mathcal{L}^*$ by

$$\blacktriangleright a_\pm = a_\pm \otimes \mathrm{e}^{\mp 2\varphi}. \tag{5.5}$$

With these actions we obtain the Hopf algebra structure of $F_z(P(1,1))$

$$\begin{array}{lll} [a_+, a_-] = -2za_- & [a_+, \varphi] = 2z(\mathrm{e}^{-\varphi} - 1) & [a_-, \varphi] = 0 \\ \Delta a_\pm = a_\pm \otimes \mathrm{e}^{\mp 2\varphi} + 1 \otimes a_\pm & \Delta\varphi = \varphi \otimes 1 + 1 \otimes \varphi & \\ \epsilon(f) = 0 & f \in \{a_\pm, \varphi\} & \\ S(a_\pm) = -a_\pm \,\mathrm{e}^{\pm\varphi} & S(\varphi) = -\varphi. & \end{array} \tag{5.6}$$

Theorem 2.1 gives easily a pair of dual bases in such a way that the duality between $U_z(\mathfrak{p}(1,1))$ and $F_z(P(1,1))$ is explicitly given by the pairing

$$\left\langle K^m P_-^n P_+^p, \varphi^q a_-^r a_+^s \right\rangle = m!n!p!\, \delta_q^m \delta_r^n \delta_s^p. \tag{5.7}$$

Now let us consider the bicrossproduct structure of $U_z(\mathfrak{p}(1,1))$ as follows

$$U_z(\mathfrak{p}(1,1)) = U(\mathfrak{k}) \triangleright\!\!\blacktriangleleft F(T_{z,2}) \tag{5.8}$$

where $\mathfrak{k}$ is the one-dimensional Lie algebra generated by K and the group $T_{z,2}$ is a deformation of the additive group $\mathbb{R}^2$ defined by the law

$$(\alpha'_-, \alpha'_+)(\alpha_-, \alpha_+) = (\alpha'_- \,\mathrm{e}^{-2z\alpha'_+}\alpha_-, \alpha'_+ + \alpha_+). \tag{5.9}$$

The functions

$$P_-(\alpha_-, \alpha_+) = \alpha_- \qquad P_+(\alpha_-, \alpha_+) = \alpha_+ \tag{5.10}$$

define a global chart on $T_{z,2}$. The $U(\mathfrak{k})$-module algebra structure of $F(T_{z,2})$ taking part in the bicrossproduct is given by

$$P_- \triangleleft K = 2P_- \qquad P_+ \triangleleft K = \frac{1}{z}(\mathrm{e}^{-2zP_+} - 1). \tag{5.11}$$

Hence, the vector field associated to K is

$$\hat{K} = 2P_- \frac{\partial}{\partial P_-} + \frac{1}{z}(\mathrm{e}^{-2zP_+} - 1)\frac{\partial}{\partial P_+}. \tag{5.12}$$

5.1.1. One-parameter flow. The vector field $\hat{K}$ has a unique equilibrium point at (0, 0), which has a hyperbolic nature. The function

$$h = P_-(\mathrm{e}^{-2zP_+} - 1) \tag{5.13}$$

is a first integral of $\hat{K}$. The computation of the integral curves requires that we solve the differential system

$$\dot{\alpha}_- = 2\alpha_- \qquad \dot{\alpha}_+ = \frac{1}{z}(\mathrm{e}^{-2z\alpha_+} - 1). \tag{5.14}$$

If $z > 0$, the integral curves placed in the region $\alpha_+ < 0$ are given by

$$\alpha_-(s) = c_1\,\mathrm{e}^{2s} \qquad \alpha_+(s) = \frac{1}{2z}\ln(1 - \mathrm{e}^{-2(s-c_2)}). \tag{5.15}$$

The second-order system associated to them is

$$\ddot{\alpha}_-(s) = 4\alpha_-(s) \qquad \ddot{\alpha}_+(s) = -\frac{2}{z}\,\mathrm{e}^{-2z\alpha_+}(\mathrm{e}^{-2z\alpha_+} - 1). \tag{5.16}$$

These equations may be interpreted as particles moving over a straight line under the action of repulsive potentials. From the expression of the integral curves we obtain the following flow

$$\Phi^s(\alpha_-, \alpha_+) = \left(\alpha_-\,\mathrm{e}^{2s}, \frac{1}{2z}\ln(1 - \mathrm{e}^{-2s}(1 - \mathrm{e}^{2z\alpha_+}))\right). \tag{5.17}$$

If, for example, we suppose that $z > 0$ then the curve that starts at the point (α_-, α_+) is defined in the interval

$$s \in \begin{cases} \left(\frac{1}{2}\ln(1 - \mathrm{e}^{2z\alpha_+}), +\infty\right) & \alpha_+ < 0 \\ (-\infty, +\infty) & \alpha_+ \geqslant 0. \end{cases} \tag{5.18}$$

Hence, the expression

$$\mathrm{e}^{sK} \triangleright (\alpha_-, \alpha_+) = \left(\alpha_-\,\mathrm{e}^{2s}, \frac{1}{2z}\ln(1 - \mathrm{e}^{-2s}(1 - \mathrm{e}^{2z\alpha_+}))\right) \tag{5.19}$$

defines a local action (in the nondeformed limit $z \to 0$ the action is global) of $\mathfrak{K}$ (the Lie group associated with the Lie algebra $\mathfrak{k}$) on $T_{z,2}$. The action decomposes $T_{z,2}$ in three strata:

(i) the point at the origin, whose isotropy group is $\mathfrak{K}$;
(ii) the four orbits constituted by the semi-axes;
(iii) the rest of the set $T_{z,2}$. This last stratum has a foliation by one-dimensional orbits: deformed hyperbolic branches.

5.1.2. Regular co-spaces. The elements of $F_z(P(1,1))$ can be written as

$$\phi(\alpha_-,\alpha_+) \qquad \phi \in F_z(\mathfrak{K}) \qquad (\alpha_-,\alpha_+) \in T_{z,2} \tag{5.20}$$

instead of the monomials $\varphi^q a_-^r a_+^s$. The expression $\phi(\alpha_-,\alpha_+)$ does not denote a function, ϕ, at the point (α_-,α_+) but the product of these two elements in the algebra $F_z(P(1,1))$.

The structure of the regular co-space $(F_z(P(1,1)), \prec, U_z(\mathfrak{p}(1,1)))$ is immediately obtained using theorem 4.2, so

$$\begin{aligned}
&(\phi(\alpha_-,\alpha_+)) \prec \mathrm{e}^{sK} = \phi(\mathrm{e}^{sK}\cdot)(\alpha_-,\alpha_+)\\
&(\phi(\alpha_-,\alpha_+)) \prec P_- = \phi\alpha_-\mathrm{e}^{2\varphi}(\alpha_-,\alpha_+)\\
&(\phi(\alpha_-,\alpha_+)) \prec P_+ = \phi\frac{1}{2z}\ln(1-\mathrm{e}^{-2\varphi}(1-\mathrm{e}^{2z\alpha_+}))(\alpha_-,\alpha_+).
\end{aligned} \tag{5.21}$$

The action on $(F_z(P(1,1)), \succ, U_z(\mathfrak{p}(1,1)))$ is given by

$$\begin{aligned}
&\mathrm{e}^{sK} \succ (\phi(\alpha_-,\alpha_+)) = \phi(\cdot\,\mathrm{e}^{sK})\left(\alpha_-\mathrm{e}^{2s}, \frac{1}{2z}\ln(1-\mathrm{e}^{-2s}(1-\mathrm{e}^{2z\alpha_+}))\right)\\
&P_- \succ (\phi(\alpha_-,\alpha_+)) = \alpha_-\phi(\alpha_-,\alpha_+)\\
&P_+ \succ (\phi(\alpha_-,\alpha_+)) = \alpha_+\phi(\alpha_-,\alpha_+).
\end{aligned} \tag{5.22}$$

In the above expressions, the dot represents the argument of the function $\phi = \phi(\cdot)$, and φ denotes the natural coordinate function in the group $\mathfrak{K}$.

Note that the elements $(\alpha_-,\alpha_+) \in T_{z,2}$ describe the subalgebra of $F_z(P(1,1))$ generated by a_- and a_+. The pair (α_-,α_+) is an eigenvector of the endomorphisms associated with the action (5.22) of the generators P_- and P_+. This fact, together with the action of $\mathfrak{K}$ on $T_{z,2}$, guarantees that the subalgebra generated by a_- and a_+ is stable under the action (5.22).

5.1.3. Induced representations. The representation of $U_z(\mathfrak{p}(1,1))$ induced by the character $(\alpha_-,\alpha_+) \in T_{z,2}$ is given according to theorem 4.3 by the following expressions

$$\phi \dashv K = \phi' \qquad \phi \dashv P_- = \phi\alpha_-\,\mathrm{e}^{2\varphi} \qquad \phi \dashv P_+ = \phi\frac{1}{2z}\ln(1-\mathrm{e}^{-2\varphi}(1-\mathrm{e}^{2z\alpha_+})). \tag{5.23}$$

Choosing a representative in each orbit, we obtain a representative of every equivalence classes of induced representations. For instance, the representation induced by the equilibrium point $(0,0) \in T_{z,2}$ is

$$\phi \dashv K = \phi' \qquad \phi \dashv P_\mp = 0. \tag{5.24}$$

The local representations induced by the character

$$K^m \vdash 1 = c^m \tag{5.25}$$

of the subalgebra $U(\mathfrak{k})$ are given, according to equation (4.40), by

$$\begin{aligned}
&\mathrm{e}^{sK} \vdash (\alpha_-,\alpha_+) = \mathrm{e}^{sc}\left(\alpha_-\,\mathrm{e}^{2s}, \frac{1}{z}\ln(1-\mathrm{e}^{-2s}(1-\mathrm{e}^{2z\alpha_+}))\right)\\
&P_- \vdash (\alpha_-,\alpha_+) = \alpha_-(\alpha_-,\alpha_+)\\
&P_+ \vdash (\alpha_-,\alpha_+) = \alpha_+(\alpha_-,\alpha_+).
\end{aligned} \tag{5.26}$$

5.2. Non-standard quantum Galilei algebra

The non-standard quantum Galilei algebra $U_z(\mathfrak{g}(1,1))$ is isomorphic to the quantum Heisenberg algebra $H_q(1)$ [36, 37] and to the deformed Heisenberg–Weyl algebra $U_\rho(HW)$

[38]. This can be obtained by contraction [38] of a non-standard deformation of the Poincaré algebra [34] (the null-plane quantum Poincaré).

The deformed Hopf algebra $U_z(\mathfrak{g}(1,1))$ has the following structure

$$\begin{array}{lll} [H,K] = -\dfrac{1-e^{-4zP}}{4z} & [P,K]=0 & [H,P]=0 \\ \Delta P = P\otimes 1 + 1\otimes P & \Delta X = X\otimes 1 + e^{-2zP}\otimes X & X\in\{H,K\} \\ \epsilon(X)=0 & X\in\{H,P,K\} & \\ S(P)=-P & S(X) = -e^{2zP}X & X\in\{H,K\}. \end{array} \tag{5.27}$$

In [35] it was proven that $U_z(\mathfrak{g}(1,1))$ has a bicrossproduct structure

$$U_z(\mathfrak{g}(1,1)) = \mathcal{K} \triangleright\!\blacktriangleleft \mathcal{L}$$

where $\mathcal{L}$ is the commutative and non-cocommutative Hopf subalgebra $U_z(\mathfrak{t}_2)$ generated by P and H, and $\mathcal{K}$ is the commutative and cocommutative Hopf algebra (it is not a Hopf subalgebra of $U_z(\mathfrak{g}(1,1))$) generated by K.

The right action of $\mathcal{K}$ on $\mathcal{L}$ is given by

$$P \triangleleft K = [P,K] = 0 \qquad H \triangleleft K = [H,K] = -\frac{1-e^{-4zP}}{4z}. \tag{5.28}$$

The left coaction of $\mathcal{L}$ over the generator of $\mathcal{K}$ is

$$K \blacktriangleleft\!\!= e^{-2zP}\otimes K. \tag{5.29}$$

The corresponding function algebra $F_z(G(1,1))$ has a bicrossproduct structure dual to the above one

$$F_z(G(1,1)) = \mathcal{K}^* \blacktriangleright\!\triangleleft \mathcal{L}^*.$$

Let v, x and t be the generators dual to K, P and H. The action of $\mathcal{L}^*$ on $\mathcal{K}^*$ is

$$x \triangleright v = -2zv \qquad t \triangleright v = 0 \tag{5.30}$$

and the coaction of $\mathcal{K}^*$ over the generators of $\mathcal{L}^*$ is

$$x \blacktriangleleft\!\!= 1\otimes x \qquad t \blacktriangleleft\!\!= 1\otimes t. \tag{5.31}$$

Action and coaction allow us to obtain the Hopf algebra structure of $F_z(G(1,1))$

$$\begin{array}{lll} [t,v]=0 & [x,v]=-2zv & [t,x]=2zt \\ \Delta t = t\otimes 1 + 1\otimes t & \Delta x = x\otimes 1 + 1\otimes x - t\otimes v & \Delta v = v\otimes 1 + 1\otimes v \\ \epsilon(f)=0 & f\in\{t,x,v\} & \\ S(v)=-v & S(x) = -x-tv & S(t)=-t. \end{array} \tag{5.32}$$

The non-degenerate pairing between $U_z(\mathfrak{g}(1,1))$ and $F_z(G(1,1))$ is given by

$$\langle K^m H^n P^p, v^q t^r x^s\rangle = m!n!p!\,\delta_q^m\delta_r^n\delta_s^p. \tag{5.33}$$

In [1] we constructed the induced representations of $U_z(\mathfrak{g}(1,1))$, however now we recover the same results but making use of its bicrossproduct structure

$$U_z(\mathfrak{g}(1,1)) = U(\mathfrak{v}) \triangleright\!\blacktriangleleft F(T_{z,2}) \tag{5.34}$$

where $\mathfrak{v}$ is the Lie algebra of the one-dimensional Galilean boosts group and $T_{z,2}$ is a deformation of the additive group $\mathbb{R}^2$ defined by

$$(b',a')(b,a) = (b' + e^{-2za'}b, a'+a). \tag{5.35}$$

In this definition we have assumed that the deformation parameter is real. Note that the composition law (5.35) is obtained from the expression of the coproduct (5.27). The elements of $T_{z,2}$ can be factorized as $(b, a) = (b, 0)(0, a)$. The coordinates on $T_{z,2}$ will be denoted by H and P, so

$$H(b, a) = b \qquad P(b, a) = a. \tag{5.36}$$

The $U(\mathfrak{v})$-module algebra $F(T_{z,2})$ is described by the action

$$H \triangleleft K = -\frac{1}{4z}(1 - \mathrm{e}^{-4zP}) \qquad P \triangleleft K = 0. \tag{5.37}$$

The vector field associated with this action on $T_{z,2}$ is

$$\hat{K} = -\frac{1}{4z}(1 - \mathrm{e}^{-4zP})\frac{\partial}{\partial H}. \tag{5.38}$$

5.2.1. One-parameter flow. Let us observe that the vector field $\hat{K}$ has infinite fixed points ($(b, 0)$, $b \in \mathbb{R}$), and P is an invariant. The integral curves

$$\dot{b} = -\frac{1}{4z}(1 - \mathrm{e}^{-4za}) \qquad \dot{a} = 0 \tag{5.39}$$

determine the autonomous system

$$b(s) = -\frac{1}{4z}(1 - \mathrm{e}^{-4zc_1})s + c_2 \qquad a(s) = c_1. \tag{5.40}$$

The flow associated to the vector field $\hat{K}$, deduced from its integral curves, is

$$\Phi^s(b, a) = \left(b - \frac{1}{4z}(1 - \mathrm{e}^{-4za})s, a\right). \tag{5.41}$$

It is defined for any value of s, giving a global action of $\mathfrak{V}$ (the Lie group associated to $\mathfrak{v}$) on $T_{z,2}$

$$\mathrm{e}^{sK} \triangleright (b, a) = \left(b - \frac{1}{4z}(1 - \mathrm{e}^{-4za})s, a\right). \tag{5.42}$$

The group $T_{z,2}$ is decomposed in two strata under this action:

(i) The set of points $(b, 0)$, each of which is an orbit with stabilizer group $\mathfrak{V}$.
(ii) The other stratum, constituted by the remaining elements of $T_{z,2}$, is a foliation with sheets $\mathcal{O}_a = \{(b, a) | a \in \mathbb{R}^*, b \in \mathbb{R}\}$. The isotropy group of the point $(0, a) \in \mathcal{O}_a$ is $\{e\}$.

5.2.2. Regular co-spaces. Theorem 4.2 determines the regular co-spaces in a direct and immediate way. Remember that $F_z(G(1, 1))$ can be described considering elements of the form

$$\phi(b, a) \qquad \phi \in F(\mathfrak{V}) \qquad (b, a) \in T_{z,2} \tag{5.43}$$

instead of the monomial elements $v^q t^r x^s$.

For $(F_z(G(1, 1)), \prec, U_z(\mathfrak{g}(1, 1)))$ we have

$$\begin{aligned} &(\phi(b, a)) \prec \mathrm{e}^{sK} = \phi(\mathrm{e}^{sK}\cdot)(b, a) \\ &(\phi(b, a)) \prec H = \phi\left(b - \tfrac{1}{4z}(1 - \mathrm{e}^{-4za})v\right)(b, a) \\ &(\phi(b, a)) \prec P = \phi a(b, a) \end{aligned} \tag{5.44}$$

and for $(F_z(G(1,1)), \succ, U_z(\mathfrak{g}(1,1)))$

$$\begin{aligned} &\mathrm{e}^{sK} \succ (\phi(b,a)) = \phi(\cdot\,\mathrm{e}^{sK})\left(b - \frac{1}{4z}(1-\mathrm{e}^{-4za})s, a\right) \\ &H \succ (\phi(b,a)) = b\phi(b,a) \\ &P \succ (\phi(b,a)) = a\phi(b,a). \end{aligned} \tag{5.45}$$

The elements $(b,a) \in T_{z,2}$ describe the subalgebra of $F_z(G(1,1))$ generated by t and x which, as in the previous case, is stable under the action (5.45).

5.2.3. Induced representations. A representative of each equivalence class of induced representations, obtained according to the theorem 4.3, is:

(i) Considering the character given by $(b, 0)$:

$$\phi \dashv \mathrm{e}^{sK} = \phi(\mathrm{e}^{sK}\cdot) \qquad \phi \dashv H = \phi b \qquad \phi \dashv P = 0. \tag{5.46}$$

(ii) Taking the character associated with $(0, a)$ the induced representation is

$$\phi \dashv \mathrm{e}^{sK} = \phi(\mathrm{e}^{sK}\cdot) \qquad \phi \dashv H = \phi\frac{-1}{4z}(1-\mathrm{e}^{-4za})v \qquad \phi \dashv P = \phi a. \tag{5.47}$$

The local representations induced by the character of $U(\mathfrak{so}_0(2))$ given by

$$K^m \vdash 1 = c^m \tag{5.48}$$

are obtained applying the result (4.40):

$$\begin{aligned} &\mathrm{e}^{sK} \vdash (b,a) = \mathrm{e}^{sc}\left(b - \frac{1}{4z}(1-\mathrm{e}^{-4za})s, a\right) \\ &H \vdash (b,a) = b(b,a) \\ &P \vdash (b,a) = a(b,a). \end{aligned} \tag{5.49}$$

The actions of the generators in the way that they were presented in [1] can be easily deduced from these expressions.

5.3. Quantum κ-Galilei algebra

A contraction of the quantum algebra $U_q(su(2))$ gives the deformation $U_\kappa(\mathfrak{g}(1,1))$ of the enveloping Galilei algebra in (1 + 1) dimensions [39]. This quantum algebra is characterized by the following commutation relations and structure mappings:

$$\begin{array}{lll} [H,K] = -P & [P,K] = \frac{P^2}{2\kappa} & [H,P] = 0 \\ \Delta H = H\otimes 1 + 1\otimes H & \Delta X = X\otimes 1 + \mathrm{e}^{-H/\kappa}\otimes X & X\in\{P,K\} \\ \epsilon(X) = 0 & X\in\{H,P,K\} & \\ S(H) = -H & S(X) = -\mathrm{e}^{H/\kappa}X & X\in\{P,K\}. \end{array} \tag{5.50}$$

The bicrossproduct structure of $U_\kappa(\mathfrak{g}(1,1))$ is

$$U_\kappa(\mathfrak{g}(1,1)) = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$$

with $\mathcal{L}$ the commutative and non-cocommutative Hopf subalgebra $U_\kappa(\mathfrak{t}_2)$ spanned by P and H, and $\mathcal{K}$ the commutative and cocommutative Hopf subalgebra generated by K (it is not a Hopf subalgebra of $U_\kappa(\mathfrak{g}(1,1))$). The right action of $\mathcal{K}$ on $\mathcal{L}$ is given by

$$P \triangleleft K = [P,K] = \frac{P^2}{2\kappa} \qquad H \triangleleft K = [H,K] = -P \tag{5.51}$$

and the left coaction of $\mathcal{L}$ over the generator of $\mathcal{K}$ is

$$K \blacktriangleleft = \mathrm{e}^{-H/\kappa} \otimes K. \tag{5.52}$$

The dual algebra has also a bicrossproduct structure

$$F_\kappa(G(1,1)) = \mathcal{K}^* \blacktriangleright\!\!\triangleleft \mathcal{L}^*$$

where $\mathcal{K}^*$ is generated by v and $\mathcal{L}^*$ by x and t. The left action of $\mathcal{L}^*$ on $\mathcal{K}^*$ is defined by

$$x \triangleright v = \frac{v^2}{2\kappa} \qquad t \triangleright v = -v/\kappa \tag{5.53}$$

and the right coaction of $\mathcal{K}^*$ on $\mathcal{L}^*$ is

$$\blacktriangleright t = t \otimes 1 \qquad \blacktriangleright x = x \otimes 1 - t \otimes v. \tag{5.54}$$

From the above action and coaction we recover the Hopf algebra structure of $F_\kappa(G(1,1))$:

$$\begin{array}{lll} [t,x] = -x/\kappa & [x,v] = \frac{v^2}{2\kappa} & [t,v] = -v/\kappa \\ \Delta t = t \otimes 1 + 1 \otimes t & \Delta x = x \otimes 1 + 1 \otimes x - t \otimes v & \Delta v = v \otimes 1 + 1 \otimes v \\ \epsilon(f) = 0 & f \in \{v,t,x\} & \\ S(v) = -v & S(x) = -x - tv & S(t) = -t. \end{array} \tag{5.55}$$

The pairing between $U_\kappa(\mathfrak{g}(1,1))$ and $F_\kappa(G(1,1))$ is now given by

$$\langle K^m P^n H^p, v^q x^r t^s \rangle = m!n!p!\,\delta_q^m \delta_r^n \delta_s^p. \tag{5.56}$$

Let us interpret the algebraic bicrossproduct structure of the quantum κ-Galilei algebra as

$$U_\kappa(\mathfrak{g}(1,1)) = U(\mathfrak{v}) \triangleright\!\!\blacktriangleleft F(T_{\kappa,2}) \tag{5.57}$$

where $T_{\kappa,2}$ is the group, deformation of the additive $\mathbb{R}^2$ group, defined by

$$(a',b')(a,b) = (a' + \mathrm{e}^{-b'/\kappa} a, b' + b). \tag{5.58}$$

The elements (a,b) can be factorized in the form $(a,b) = (a,0)(0,b)$. The functions P and H defined by

$$P(a,b) = a \qquad H(a,b) = b \tag{5.59}$$

determine a global chart on $T_{\kappa,2}$.

The action of the generator K on the $U(\mathfrak{v})$-module algebra $F(T_{\kappa,2})$ is given by

$$H \triangleleft K = -P \qquad P \triangleleft K = \frac{P^2}{2\kappa}. \tag{5.60}$$

Hence, the induced vector field is

$$\hat{K} = \frac{P^2}{2\kappa}\frac{\partial}{\partial P} - P\frac{\partial}{\partial H}. \tag{5.61}$$

5.3.1. One-parameter flow. The invariant points of the vector field $\hat{K}$ are $(0,b)$. To obtain an invariant function under the action of $\hat{K}$ is sufficient to determine firstly the one-forms, η, verifying $\hat{K} \rfloor \eta = 0$. The general solution is

$$\eta_\alpha = \alpha\left(\mathrm{d}P + \frac{1}{2\kappa} P\,\mathrm{d}H\right) \qquad \alpha \in F(T_{\kappa,2}). \tag{5.62}$$

Choosing $\alpha_0 = 1/P$, the one-form η_{α_0} is exact and invariant under $\hat{K}$. So, the invariant function is $h = P\,\mathrm{e}^{H/2\kappa}$. The autonomous system

$$\dot{a} = \frac{a^2}{2\kappa} \qquad \dot{b} = -a \tag{5.63}$$

which determines the integral curves, is easily integrated. For the curves placed in the region $a < 0$ we find the following expressions:

$$a(s) = \frac{-1}{c_1 + \frac{s}{2\kappa}} \qquad b(s) = 2\kappa \ln\left(c_1 + \frac{s}{2\kappa}\right) + c_2. \tag{5.64}$$

The associated second-order equations

$$\ddot{a} - \frac{a^3}{2\kappa^2} = 0 \qquad \ddot{b} + \frac{\dot{b}^2}{2\kappa} = 0 \tag{5.65}$$

can be interpreted as particles moving in a straight line under forces depending on the position or the velocity, respectively. From the expression of the integral curves the flow associated with $\hat{K}$ is obtained:

$$\Phi^s(a, b) = \left(\frac{a}{1 - \frac{sa}{2\kappa}}, b + 2\kappa \ln\left(1 - \frac{sa}{2\kappa}\right)\right). \tag{5.66}$$

Note that the action of $\mathfrak{V}$ on $T_{\kappa,2}$,

$$\mathrm{e}^{sK} \triangleright (a, b) = \left(\frac{a}{1 - \frac{sa}{2\kappa}}, b + 2\kappa \ln\left(1 - \frac{sa}{2\kappa}\right)\right) \tag{5.67}$$

is not global. The space $T_{\kappa,2}$ is decomposed in two strata under this action:

(i) The set points of the form $(0, b)$. Each of them constitutes a zero-dimensional orbit with stabilizer $\mathfrak{V}$.
(ii) The other stratum, constituted by the rest of the space, presents a foliation by one-dimensional sheets.

5.3.2. Regular co-spaces. The action on the regular co-space $(F_\kappa(G(1, 1)), \prec, U_\kappa(\mathfrak{g}(1, 1)))$ is obtained applying theorem 4.2

$$\begin{aligned}
&(\phi(a, b)) \prec \mathrm{e}^{sK} = \phi(\mathrm{e}^{sK}\cdot)(a, b)\\
&(\phi(a, b)) \prec P = \phi\frac{a}{1 - \frac{av}{2\kappa}}(a, b)\\
&(\phi(a, b)) \prec H = \phi\left(b + 2\kappa \ln\left(1 - \frac{av}{2\kappa}\right)\right)(a, b)
\end{aligned} \tag{5.68}$$

with $\phi \in F(\mathfrak{V})$ and $(a, b) \in T_{\kappa,2}$.

The co-space $(F_\kappa(G(1, 1)), \succ, U_\kappa(\mathfrak{g}(1, 1)))$ is analogously described by

$$\begin{aligned}
&\mathrm{e}^{sK} \succ (\phi(a, b)) = \phi(\cdot\, \mathrm{e}^{sK})\left(\frac{a}{1 - \frac{as}{2\kappa}}, b + 2\kappa \ln\left(1 - \frac{as}{2\kappa}\right)\right)\\
&P \succ (\phi(a, b)) = a\phi(a, b)\\
&H \succ (\phi(a, b)) = b\phi(a, b).
\end{aligned} \tag{5.69}$$

Similar comments to those of subsections 5.1.2 and 5.2.2 may be made here.

5.3.3. Induced representations. According to theorem 4.3 each element $(a, b) \in T_{\kappa,2}$ induces a representation given by

$$\begin{aligned} \phi \dashv \mathrm{e}^{sK} &= \phi(\mathrm{e}^{sK}\cdot) \\ \phi \dashv P &= \phi \frac{a}{1-\frac{av}{2\kappa}} \\ \phi \dashv H &= \phi\left(b + 2\kappa \ln\left(1 - \frac{av}{2\kappa}\right)\right) \end{aligned} \tag{5.70}$$

which effectively coincides with what was obtained in [1].

The local representations induced by the character of $U(\mathfrak{v})$ given by

$$K^m \vdash 1 = c^m \tag{5.71}$$

are constructed by applying the result (4.40)

$$\begin{aligned} \mathrm{e}^{sK} \vdash (a, b) &= \mathrm{e}^{sc}\left(\frac{a}{1-\frac{as}{2\kappa}}, b + 2\kappa \ln\left(1 - \frac{as}{2\kappa}\right)\right) \\ P \vdash (a, b) &= a(b, a) \\ H \vdash (a, b) &= b(b, a). \end{aligned} \tag{5.72}$$

From these expressions the actions of the generators are easily obtained.

6. Concluding remarks

In [1] we introduced an algebraic method for constructing (co)induced representations of Hopf algebras based on the existence of a triplet composed by two Hopf algebras and a non-degenerate pairing between them such that there exists a pairing of dual bases. However, the difficulty of the computation of the normal ordering of a product of elements increases with the number of algebra generators. In this paper, we avoid these problems when the quantum algebra has a bicrossproduct structure.

We are able to define structures over a bicrossproduct Hopf algebra $H = K \triangleright\!\!\blacktriangleleft L$ in terms of those of its components K and L. So, theorem 2.1 gives a procedure to obtain dual bases of the pair (H, H^*) starting from the dual bases of the components. Analogously, theorem 2.2 characterizes a $*$-structure for the algebra sector of H from the $*$-structures defined on K and L.

Our induction procedure is not a generalization of the induction method for Lie groups. We introduce the concept of co-space, which generalizes in an algebraic way the concept of G-space (with G being a transformation group), and we establish the connection between induced representations and regular co-spaces. There are different procedures to compute regular co-spaces but the introduction of the endomorphisms associated with the regular actions and the use of adjoint operators with respect to the duality form simplifies extraordinarily the computations [1]. Note that vector fields have been used to compute commutators and it is advisable to use exponential elements instead of monomial bases. For bicrossproduct Hopf algebras, such as $H = K \triangleright\!\!\blacktriangleleft L$, with K cocommutative and L commutative, theorems 3.1 and 3.2 establish a connection between the representations of H, induced by characters of L, and certain one-parameter flows. Although the proof is based on the use of pairs of dual bases the results so obtained are, essentially, independent of the bases used. Moreover, we can associate, in some sense, quantum bicrossproduct groups with dynamical systems via these flows. This relation has been analysed in more detail in [18].

The bicrossproduct Hopf algebras such as $H = \mathcal{K} \triangleright\!\!\blacktriangleleft \mathcal{L}$, ($\mathcal{K}$ and $\mathcal{L}$ are commutative and cocommutative, respectively, infinite-dimensional algebras), have been studied interpreting $\mathcal{K}$ as the enveloping algebra $U(\mathfrak{k})$ of a Lie group K and $\mathcal{L}$ as the algebra of functions over a Lie group L. From this point of view, certain families of Hopf algebras that are deformations of semi-direct products can be seen as homotopical deformations of the original actions.

The description of the regular co-spaces associated with H may be carried out without monomial bases. Theorem 4.1 proves that the action on such co-spaces may be obtained using the action, deduced from the bicrossproduct structure, of the group K on the group L. In this way, the problems derived from the use of dual bases and the high dimension of the algebra H are avoided.

The description of the (induced) representations appears as a corollary of the above-mentioned theorems. The bicrossproduct algebra $H = U(\mathfrak{k}) \triangleright\!\!\blacktriangleleft F(L)$ gives, in a natural way, a $*$-structure for which the representations are, essentially, unitary. Theorem 4.3 discusses the equivalence of the induced representations establishing a correspondence among classes of induced representations and orbits of L under the action of K. This result is in some sense analogous to the Kirillov orbits method [13]. The problem of the irreducibility of the representations is still open. Partial results for particular cases have been obtained; for instance, see [14] for the standard quantum (1 + 1) Galilei algebra and [26] for the quantum extended (1 + 1) Galilei algebra.

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